Lesson 17. Profit Maximization

1 Overview

- In Lesson 16, we saw an example of a profit maximization problem
  - A company produces and sells two products
  - Prices are exogenously determined
  - How much of each product should the company produce in order to maximize its profit?
- This lesson: what if the company is free to charge whatever price they wish?

2 Incorporating demand into profit maximization

- Consider a company that produces and sells three products
- The company can set the price of these products
- The demand for these products depends on their prices
- Variables:

R = revenue	Q_1 = quantity of product 1 produced and sold	P_1 = unit price of product 1
C = cost	Q_2 = quantity of product 2 produced and sold	P_2 = unit price of product 2
	Q_3 = quantity of product 3 produced and sold	P_3 = unit price of product 3

- Model:

$$\begin{aligned} \text{maximize} \quad & R - C \\ \text{subject to} \quad & R = P_1 Q_1 + P_2 Q_2 + P_3 Q_3 \\ & C = 20 + 15(Q_1 + Q_2 + Q_3) \\ & Q_1 = \frac{63}{4} - \frac{1}{4} P_1 \\ & Q_2 = 21 - \frac{1}{5} P_2 \\ & Q_3 = \frac{25}{2} - \frac{1}{6} P_3 \end{aligned}$$

- Let's determine what the company needs to produce and sell in order to maximize profit

Step 0. Simplify the model

- First, let's simplify the model by <u>expressing profit π as a function of Q_1, Q_2 and Q_3</u>
- Let's start by solving for P_i in terms of Q_i ($i = 1, 2, 3$):

- Now we can express R as a function of Q_1, Q_2, Q_3 by substitution:

- Next, we can express profit π as a function of Q_1, Q_2, Q_3 by substitution as well:

- Now, let's maximize π

Step 1. Find the critical points

- The gradient of π is

- The first-order necessary condition tells us that critical points of π must satisfy

- Therefore, we have one critical point of π:

Step 2. Classify each critical point as a local minimum, local maximum, or saddle point

- The Hessian matrix of π is

- The Hessian matrix of π at the critical point $(Q_1, Q_2, Q_3) = (6, 9, 5)$ is

- The principal minors of the Hessian at $(Q_1, Q_2, Q_3) = (6, 9, 5)$ are

- Therefore, the second derivative test tells us that

- So, the company's locally optimal production plan and profit is:

3 Exercises

Problem 1. Supppose we have a company that manufactures two products that are sold in the same market. The company has a monopoly and may charge whatever prices it wishes. Let

R = revenue	Q_1 = quantity of product 1 produced and sold	P_1 = unit price of product 1
C = cost	Q_2 = quantity of product 2 produced and sold	P_2 = unit price of product 2

Assume that the demand of the two products depends on their prices as follows:

$$Q_1 = 40 - 2P_1 + P_2$$
$$Q_2 = 15 + P_1 - P_2$$

In addition, assume the cost of production is $C = Q_1^2 + Q_1Q_2 + Q_2^2$. How much of each product should the company manufacture in order to maximize total profit?