Lesson 18. Optimization with Equality Constraints

1 The effect of a constraint

- Let's model a consumer whose utility depends on his or her consumption of two products
- Define the following variables:

x_1 = units of product 1 consumed $\quad\quad$ x_2 = units of product 2 consumed

- The consumer's utility function is

$$f(x_1, x_2) = x_1x_2 + 2x_1 + 2x_2$$

- Without any additional information, the consumer can maximize his or her utility by

- To make this model more realistic, we should take into account the consumer's budget
- Suppose the unit prices of products 1 and 2 are \$1 and \$3 respectively
- In addition, suppose the consumer intends to spend \$10 on the two products
- The consumer's budget constraint can be expressed as

- Putting this all together, we obtain the following optimization model:

$$\begin{aligned} \text{maximize} \quad & x_1x_2 + 2x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 3x_2 = 10 \end{aligned}$$

- We have seen models like this before, with an **objective function** to be maximized/minimized, and **equality constraints** defining relationships between the variables — e.g. profit maximization
- Sometimes we can solve these models by first substituting the equality constraint into the objective function, and then finding the minimum/maximum of the resulting objective function
- This isn't always possible, especially when the equality constraint is complex
- Instead, we can use **the method of Lagrange multipliers**

2 The Lagrange multiplier method – 1 equality constraint

$$\begin{aligned} \text{minimize/maximize} \quad & f(x_1, \ldots, x_n) \\ \text{subject to} \quad & g(x_1, \ldots, x_n) = c \end{aligned}$$

- The **Lagrangian function** L is

$$L(\lambda, x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \lambda\big[g(x_1, \ldots, x_n) - c\big]$$

- The gradient of L is

- The Hessian of L (also known as the **bordered Hessian**) is:

Finding constrained local optima:

- **Step 0.** Form the Lagrangian function L and find its gradient and Hessian
- **Step 1.** Find the **constrained critical points** $(\lambda, x_1, \ldots, x_n)$ that solve the following system of equations:

$$\nabla L(\lambda, x_1, \ldots, x_n) = 0 \quad \text{or equivalently} \quad \begin{aligned} g(x_1, \ldots, x_n) &= c \\ \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n) &= \lambda \frac{\partial g}{\partial x_1}(x_1, \ldots, x_n) \\ &\vdots \\ \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n) &= \lambda \frac{\partial g}{\partial x_n}(x_1, \ldots, x_n) \end{aligned}$$

- **Step 2.** Classify each constrained critical point as a local minimum, local maximum, or saddle point by applying the **second derivative test for constrained extrema**:
    - Suppose $(\lambda^*, x_1^*, \dots, x_n^*)$ is a constrained critical point found in Step 1
    - Compute the principal minors $d_i = |H_L(\lambda^*, x_1^*, \dots, x_n^*)|$ for $\underline{i = 3, \dots, n+1}$
    - If $d_{n+1} \neq 0$:

(1) $-d_3 > 0, \dots, -d_{n+1} > 0$	then f has a constrained local minimum at $(x_1^*, \dots, x_n^*)$
(2) $-d_3 < 0, -d_4 > 0, -d_5 < 0, \dots$	then f has a constrained local maximum at $(x_1^*, \dots, x_n^*)$
(3) otherwise,	f has a constrained saddle point at $(x_1^*, \dots, x_n^*)$

    - If $d_{n+1} = 0$, then the test gives no information

Example 1. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & x_1 x_2 + 2x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 3x_2 = 10 \end{aligned}$$

Step 0. Form the Lagrangian function L and find its gradient and Hessian.

Step 1. Find the constrained critical points.

Step 2. Classify each constrained critical point as a local minimum, local maximum, or saddle point by applying the second derivative test for constrained extrema.

Example 2. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} \quad & 2x_1 + x_2 + 4x_3 = 168 \end{aligned}$$