Lesson 21. Optimization with Inequality Constraints

1 Last time...

- Optimization with equality constraints using Lagrange multipliers
- What if the constraints are inequalities?

2 General strategy

$$\begin{aligned} \text{minimize/maximize} \quad & f(x_1, \dots, x_n) \\ \text{subject to} \quad & g_1(x_1, \dots, x_n) \leq c_1 \\ & \vdots \\ & g_k(x_1, \dots, x_n) \leq c_k \end{aligned}$$

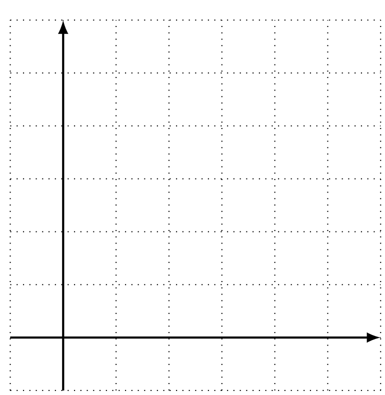
- The set of points $(x_1, \dots, x_n)$ that satisfy the constraints is called the **feasible region**
- If the feasible region is closed and bounded, then the following strategy works:
  1. Find the values at the unconstrained critical points of f.
  2. Find the extreme values of f on the boundary of the feasible region.
  3. The largest of the values from Steps 1 and 2 is the absolute maximum value.
     The smallest of these values is the absolute minimum value.

Example 1. Suppose we have \$5 to spend on apples and bananas. If we spend x on apples and y on bananas, our utility is $f(x, y) = 10 - x^2 - y^2 - 2x + 2y$. We want to maximize our utility.

Our model is

$$\begin{aligned} \text{maximize} \quad & 10 - x^2 - y^2 - 2x + 2y \\ \text{subject to} \quad & x + y \leq 5 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

Draw the feasible region:



From our drawing, it is clear that the feasible region is closed and bounded.

Find the unconstrained critical points of f. Which ones are in the interior of the feasible region?

Now let's consider the boundaries of the feasible region. First, the vertical line segment from $(0,0)$ to $(0,5)$.

Next, the horizontal line segment from $(0,0)$ to $(5,0)$.

Finally, the line segment between $(0, 5)$ and $(5, 0)$.

Putting this all together: what is our maximum possible utility? How many apples and bananas should we buy?

3 Exercises

Example 2. Simplexville Industries manufactures and sells 2 products. When the company manufactures and sells Q_1 units of product 1 and Q_2 units of product 2, its profit is

$$\pi(Q_1, Q_2) = 5Q_1 - \frac{1}{3}Q_1^2 + 4Q_2 - \frac{1}{2}Q_2^2.$$

The company can only manufacture nonnegative amounts of product ($Q_1 \geq 0$, $Q_2 \geq 0$). In addition, each unit of product 1 requires 1 unit of raw material, each unit of product 2 requires 2 units of raw material, and the company has 16 units of raw material available ($Q_1 + 2Q_2 \leq 16$). Determine how much product 1 and product 2 the company should manufacture in order to maximize its profit.