Lesson 22. A Very Brief Introduction to Linear Programming

1 The Farmer Jones problem

Farmer Jones decides to supplement the farm's income by baking and selling two types of cakes, chocolate and vanilla. Each chocolate cake sold gives a profit of \$3, and the profit on each vanilla cake sold is \$4. Each chocolate cake uses 4 eggs and 4 pounds of flour, while each vanilla cake uses 2 eggs and 6 pounds of flour. If Farmer Jones has only 32 eggs and 48 pounds of flour available, how many of each type of cake should Farmer Jones bake in order to maximize her profit? (For now, assume all cakes baked are sold, and fractional cakes are OK.)

- Let x be a variable that represents the number of chocolate cakes Farmer Jones bakes
- Similarly, let y be a variable that represents the number of vanilla cakes Farmer Jones bakes
- Objective function – what are we trying to minimize or maximize?
- Constraints – what are the allowable values of x and y?
- Putting this all together, we have the following **linear program**:

$$\begin{array}{rl} \text{maximize} & 3x + 4y \\ \text{subject to} & 4x + 2y \leq 32 \\ & 4x + 6y \leq 48 \\ & x \geq 0 \\ & y \geq 0 \end{array}$$

2 What makes a linear program?

- We classify optimization problems based on characteristics of
  - variables
  - constraints
  - objective function

- Variables can be continuous or integral
  - **Continuous**: can take on any value in a specified interval, e.g. $[0, +\infty)$
  - **Integral**: restricted to a specified interval of integers, e.g. $\{0, 1\}$
- Functions can be linear or nonlinear
  - A function $f(x_1, \dots, x_n)$ is **linear** if it is a constant-weighted sum of $x_1, \dots, x_n$; i.e.

$$f(x_1, \dots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

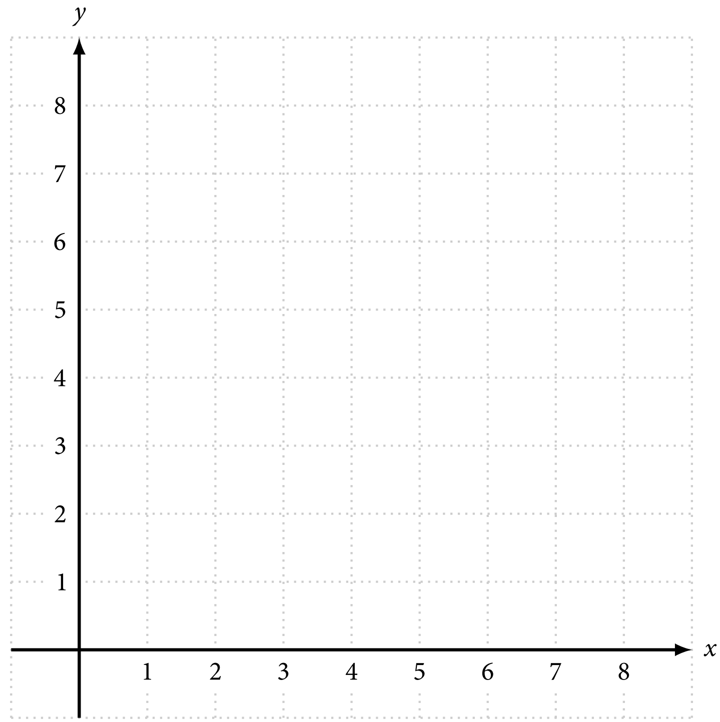
    where $c_1, \dots, c_n$ are constants
  - Otherwise, a function is **nonlinear**
- Are these functions linear or nonlinear?
  - $f(x_1, x_2, x_3) = 9x_1 - 17x_3$
  - $f(x_1, x_2, x_3) = \frac{5}{x_1} + 3x_2 - 6x_3$
  - $f(x_1, x_2, x_3) = \frac{x_1 - x_2}{x_2 + x_3}$
  - $f(x_1, x_2, x_3) = x_1 x_2 + 3x_3$
- Constraints can be linear or nonlinear
  - A constraint can be written in the form

$$g(x_1, \dots, x_n) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} b \qquad (*)$$

    where $g(x_1, \dots, x_n)$ is a function of variables $x_1, \dots, x_n$ and b is a specified constant
  - Constraint $(*)$ is **linear** if $g(x_1, \dots, x_n)$ is linear and **nonlinear** otherwise
- An optimization problem is a **linear program (LP)** if
  - the variables are continuous
  - the objective function is linear, and
  - the constraints are linear
- Are these optimization problems linear programs?
  - $$\begin{aligned} \max \quad & 3z_1 + 14z_2 + 7z_3 \\ \text{s.t.} \quad & 10z_1 + 5z_2 \leq 25 - 18z_3 \\ & z_1 \geq 0, z_2 \geq 0, z_3 \geq 0 \end{aligned}$$
  - $$\begin{aligned} \min \quad & 3w_1 + 14w_2 - w_3 \\ \text{s.t.} \quad & 3w_1 + w_2 \leq 1 \\ & w_1 w_2 w_3 = 1 \\ & w_1 + 2w_2 + w_3 = 10 \\ & w_1 \geq 0, w_3 \geq 0 \\ & w_1 \text{ integer} \end{aligned}$$
  - Farmer Jones's problem
- There are other types of optimization problems: e.g. nonlinear programs, integer programs

3 Solving linear programs with two variables

- The **feasible region** – the collection of all allowable values of (x, y) – for Farmer Jones's optimization problem:



- If the feasible region of a linear program is bounded, then there must exist a globally optimal solution at a **corner point** of the feasible region
    - This result still holds when there are n variables – we can extend the idea of a "corner point" to n dimensions
    - This is the basis behind the **simplex method**
- Corner points and corresponding objective function values for Farmer Jones's problem:

- Therefore, an optimal solution of Farmer Jones's problem is:

4 Exercises

Problem 1. The Simplex Company produces 2 products, X and Y. The company charges \$20 per unit of X and \$15 per unit of Y. The market is large enough so that the company can sell all that it produces at these prices. The production of each product requires 3 inputs, denoted A, B, and C. The company has 60 units of A, 24 of B, and 84 of C available for use. To produce a single unit of X, the company needs 5 units of A, 3 units of B, and 12 units of C. To produce a single unit of Y, the company needs 15 units of A, 4 units of B, and 7 units of C. The company wants to maximize the revenue it receives from selling the two products.

a. Formulate the company's problem as a linear program.

b. Solve the linear program you wrote in part a. How much should the company produce of X and Y to maximize its revenue? What is the company's maximum possible revenue?