Discrete Dynamical Systems – Writing About Stability

In this note, we give some examples of how to write about the stability and the fixed points of a DS.

See Lesson 8 for the relevant definitions for a second-order linear DS.

General tips:

- When you take the limit of an expression that involves free constants (e.g., c_1, c_2), take care to qualify your statement in terms of the values of the free constants. Examples:
  - "$A_n \to 1$ as $n \to \infty$ for all values of c_1 and c_2"
  - "$|A_n| \to \infty$ as $n \to \infty$ when $c_1 \neq 0$"
- A DS can be classified as stable, unstable, or neither. This classification does not depend on the values of the free constants. Examples of incorrect statements:
  - "If $c_1 = 0$, then the system is stable."
  - "The system is unstable as long as $c_2 \neq 0$."

Example 1. Consider the second-order linear DS $A_{n+2} = \frac{5}{6}A_{n+1} - \frac{1}{6}A_n + 1$, $n = 0, 1, 2, \dots$. The general solution is

$$A_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n + 3$$

and the fixed point of this DS is 3. Is the system stable or unstable? Is the fixed point attracting or repelling? Briefly explain.

Solution.

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n + 3\right]$$

Since $(\frac{1}{2})^n \to 0$ and $(\frac{1}{3})^n \to 0$ as $n \to \infty$, $A_n \to 3$ as $n \to \infty$ for all values of c_1 and c_2. Therefore, $\lim_{n\to\infty} A_n$ exists for all initial conditions, which means the system is stable.

Furthermore, $A_n \to 3$ as $n \to \infty$ for all values of c_1 and c_2 and 3 is the fixed point of this DS. Therefore, the fixed point 3 is attracting.

Example 2. Consider the DS $A_{n+2} = \frac{5}{2}A_{n+1} - A_n + 2$, $n = 0, 1, 2, \dots$. The general solution is

$$A_n = c_1\left(\frac{1}{2}\right)^n + c_2 2^n + \frac{4}{5}$$

and the fixed point is -4. Is the system stable or unstable? Is the fixed point attracting or repelling? Briefly explain.

Solution.

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[c_1\left(\frac{1}{2}\right)^n + c_2 2^n + \frac{4}{5}\right]$$

Since $2^n \to \infty$ as $n \to \infty$, $|A_n| \to \infty$ as $n \to \infty$ when $c_2 \neq 0$. Therefore, there exists an initial condition for which $\lim_{n\to\infty} |A_n| = \infty$, which means the system is unstable.

Furthermore, since $|A_n| \to \infty$ as $n \to \infty$ when $c_2 \neq 0$, there exists an initial condition for which A_n does not approach the fixed point -4 as $n \to \infty$. Therefore, the fixed point -4 is repelling.

Example 3. Consider the DS $A_{n+2} = -\frac{1}{2}A_{n+1} + \frac{1}{2}A_n + 1$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1\left(\frac{1}{2}\right)^n + c_2(-1)^n + 1$$

and the fixed point is 1. Is the system stable or unstable? Is the fixed point attracting or repelling? Briefly explain.

Solution.

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[c_1\left(\frac{1}{2}\right)^n + c_2(-1)^n + 1\right]$$

Since $(\frac{1}{2})^n \to 0$ and $(-1)^n$ oscillates between -1 and 1 as $n \to \infty$, A_n will oscillate between two values ($-c_2 + 1$ and $c_2 + 1$) as $n \to \infty$ <u>when $c_2 \neq 0$</u>. Therefore, <u>there exists an initial condition</u> for which $\lim_{n\to\infty} |A_n|$ does not exist and $|A_n| \not\to \infty$ as $n \to \infty$, which means the system is neither stable nor unstable.

Furthermore, since A_n oscillates between two values <u>when $c_2 \neq 0$</u>, <u>there exists an initial condition</u> for which A_n does not approach the fixed point 1 and $|A_n| \not\to \infty$ as $n \to \infty$. Therefore, the fixed point 1 is neither attracting nor repelling.