The Leontief Input-Output Model – Summary

The setup

- n industries: industry i produces product i
- x_i = output of industry i, in dollars
- d_i = consumer demand for product i, in dollars (**final demand**)
- a_{ij} = dollars of product i required to produce one dollar of product j (**input demand**)
  - Put another way: a_{ij} = amount that industry j pays to industry i for each dollar of product j produced
- Question: How much should each industry produce in order to meet both the final and input demands?

The model

- Let's focus on the case when $n = 3$
- Let A be the matrix of input demands, let D be the vector of final demands, let X be the vector of outputs:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- The outputs X must satisfy the following system of linear equations:

$$(I - A)X = D \qquad \Leftrightarrow \qquad \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- $(I - A)$ is called the **Leontief matrix**

Solving the model

- If $(I - A)$ is invertible, then the output vector X is:

$$X = (I - A)^{-1} D$$

- Labor (**primary inputs**)
  - a_{0j} = Primary inputs for product j
    = How much industry j spends on labor for each dollar of product j produced
    = $1 - (a_{1j} + a_{2j} + a_{3j})$

  ⇒ Amount that industry j spends on labor = $a_{0j}x_j$

  ⇒ Total labor cost for all industries
    = Total required amount of primary inputs
    = $a_{01}x_1 + a_{02}x_2 + a_{03}x_3$
- Total amount that industry j pays industry i = $a_{ij}x_j$