Mathematics for Economics

1 Market Models

1.1 Linear Market Models with One Commodity

Suppose that the demand D for a product is related to the price P by the linear equation $D = a - bP$, with $a, b > 0$, and that the supply S of the product is given by $S = -c + dP$, with $c, d > 0$. Note that as the price increases, the demand decreases and the supply increases. If $D > S$, there is excess demand and the price will rise. If $D < S$, there is excess supply and the price will fall. In both cases, the price is changing. The market is said to be in an **equilibrium state** if the price is unchanging. For this to be the case, we must have $D = S$. A model for market equilibrium is therefore

$$\begin{aligned} &D = S \\ &D = a - bP,\ a, b > 0 \\ &S = -c + dP,\ c, d > 0. \end{aligned}$$

We choose $c > 0$ since it is reasonable to assume that no supply will be available until price reaches some minimum positive value.

Given this model, we would like to find the price P_e at which the market is in equilibrium. Since $D = S$ at equilibrium, we have $a - bP_e = -c + dP_e$, which can be written as

$$P_e = \frac{a+c}{b+d}.$$

If $P = P_e$, then

$$S = D = a - bP_e = a - b\left(\frac{a+c}{b+d}\right) = \frac{ad-bc}{b+d}.$$

This shows that in order to have $S = D$ be positive, we must have $ad - bc > 0$.

1.2 Nonlinear Market Models with One Commodity

It is possible that the demand and supply functions may not depend on price in a linear manner. For example, we might have $D = 8 - 3P^2$, $S = P^2 - 1$. We can find the equilibrium price for such a market model with the same technique we used in the linear case. If $D = S$, then we have

$$8 - 3P^2 = P^2 - 1,$$

so that

$$P^2 = \frac{9}{4},$$

giving $P = \frac{3}{2}$. More generally, if the model for market equilibrium is

$$\begin{aligned} D &= S \\ D &= f(P) \\ S &= g(P), \end{aligned}$$

we can find the equilibrium price by solving the equation $f(P) = g(P)$ for P.

1.3 Linear Market Models with Two Commodities

Supppose we have two commodities which are related to each other. A linear model for market equilibrium is given by

$$\begin{aligned} D_1 &= S_1 \\ D_1 &= d_0 + d_1 P_1 + d_2 P_2 \\ S_1 &= s_0 + s_1 P_1 + s_2 P_2 \\ D_2 &= S_2 \\ D_2 &= \delta_0 + \delta_1 P_1 + \delta_2 P_2 \\ S_2 &= \sigma_0 + \sigma_1 P_1 + \sigma_2 P_2, \end{aligned} \tag{1}$$

where D_i is the demand and S_i is the supply for product i, $i = 1, 2$. Setting $D_i = S_i$ for $i = 1, 2$ gives

$$\begin{aligned} d_0 + d_1 P_1 + d_2 P_2 &= s_0 + s_1 P_1 + s_2 P_2 \\ \delta_0 + \delta_1 P_1 + \delta_2 P_2 &= \sigma_0 + \sigma_1 P_1 + \sigma_2 P_2. \end{aligned}$$

This is a system of two linear equations in the two variables P_1, P_2 which is easily solved. For example, consider the model

$$\begin{aligned} D_1 &= S_1 \\ D_1 &= 7 - 4P_1 + 2P_2 \\ S_1 &= -6 + 4P_1 - P_2 \\ D_2 &= S_2 \\ D_2 &= 1 + P_1 - P_2 \\ S_2 &= -4 - P_1 + 2P_2. \end{aligned} \tag{2}$$

This model leads to the linear system

$$\begin{aligned} 7 - 4P_1 + 2P_2 &= -6 + 4P_1 - P_2 \\ 1 + P_1 - P_2 &= -4 - P_1 + 2P_2, \end{aligned}$$

which simplifies as

$$\begin{aligned} 8P_1 - 3P_2 &= 13 \\ 2P_1 - 3P_2 &= -5. \end{aligned}$$

The solution to this system is $(P_1, P_2) = (3, \frac{11}{3})$. Substituting these prices in the expressions for D_i or S_i in (2) give us the supply and demand values in the equilibrium state. We obtain $D_1 = S_1 = \frac{7}{3}$, $D_2 = S_2 = \frac{1}{3}$. Note that all these values are nonnegative. Other choices of the constants in the equations in (1) could produce negative values of the D_i or S_i, which would indicate that the model was unrealistic from an economic standpoint.

Problems for Section 1

1. Find the equilibrium price P_e for each of the following models.

a.

$$\begin{aligned} D &= S \\ D &= 3 - 2P \\ S &= -1 + 5P \end{aligned}$$

b.

$$\begin{aligned} D &= S \\ D &= 5 - 2P^2 \\ S &= -1 + 3P \end{aligned}$$

c.

$$\begin{aligned} D &= S \\ D &= 4 - P^2 \\ S &= P^2 + 2P - 8 \end{aligned}$$

2. For the model

$$\begin{aligned} D_1 &= S_1 \\ D_1 &= 5 - 2P_1 + P_2 \\ S_1 &= -3 + 5P_1 - 2P_2 \\ D_2 &= S_2 \\ D_2 &= 2 + 2P_1 - P_2 \\ S_2 &= -43 - 2P_1 + 3P_2, \end{aligned}$$

a. Find P_1 and P_2 when the market is in an equilibrium state.

b. Find the demand and supply levels D_i and S_i, $i = 1, 2$, when the market is in an equilibrium state.

2 Matrices

We have seen that linear market models lead to systems of linear equations. If there are many commodities in the model, we will have a system with many variables. Matrices are a useful tool for solving linear systems of any size.

2.1 Basic Concepts

A **matrix** is a rectangular array

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}.$$

Each entry A_{ij} can be taken to be a real number or a variable. A has m **rows** and n **columns**. The i-th row is

$$\begin{bmatrix} A_{i1} & A_{i2} & \cdots & A_{in} \end{bmatrix}$$

and the j-th column is

$$\begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{bmatrix}.$$

The matrix entry A_{ij} is in the i-th row and the j-th column. The **size** of A is $m \times n$ (read as m by n). For example,

$$A = \begin{bmatrix} 1 & 3 \\ 9 & 6 \\ 4 & 7 \end{bmatrix}$$

is a 3×2 matrix with $A_{11} = 1$, $A_{12} = 3$, etc.

A $1 \times n$ matrix is called a **row vector**. An $m \times 1$ matrix is called a **column vector**. An n-dimensional **vector** is written $\mathbf{x} = (x_1, \ldots, x_n)$. An n-dimensional vector can also be considered as a point in n-dimensional space $\mathbb{R}^n$. Row vectors and column vectors can be considered as vectors. If

$\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are vectors, then the **dot product** of $\mathbf{x}$ and $\mathbf{y}$ is the scalar

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

For example, if $\mathbf{x} = (3, 1, 4)$ and $\mathbf{y} = (2, 0, 5)$, then
$\mathbf{x} \cdot \mathbf{y} = 3 \cdot 2 + 1 \cdot 0 + 4 \cdot 5 = 26$.

2.2 Matrix Arithmetic

Two $m \times n$ matrices A and B are **equal** if $A_{ij} = B_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. If A and B are two $m \times n$ matrices, then the **sum** $A + B$ and the **difference** $A - B$ are $m \times n$ matrices, with

$$(A + B)_{ij} = A_{ij} + B_{ij} \text{ and } (A - B)_{ij} = A_{ij} - B_{ij}.$$

For example,

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 9 & 5 \end{bmatrix}.$$

If c is a scalar and A is a matrix, then cA is a matrix of the same size as A, with $(cA)_{ij} = cA_{ij}$. For example,

$$3 \begin{bmatrix} 4 & 5 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 15 \\ 24 & 6 \end{bmatrix}.$$

If A is $m \times n$ and B is $p \times q$, then the **product** AB of A and B is defined if and only if $n = p$, and then AB is $m \times q$ with

$$(AB)_{ij} = (\text{Row } i \text{ of } A) \cdot (\text{ Column } j \text{ of } B).$$

For example,

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 20 & 8 \end{bmatrix}.$$

If A is an $n \times n$ matrix and m is a positive integer, then A^m is defined to be

$$\underbrace{A \cdot A \cdots A}_{m}.$$

If $\mathbf{v}$ is a $1 \times n$ row vector and $\mathbf{w}$ is an $n \times 1$ column vector, then the matrix product $\mathbf{vw}$ is a scalar (a 1×1 matrix) which equals the dot product $\mathbf{v} \cdot \mathbf{w}$

of $\mathbf{v}$ and $\mathbf{w}$ (considered as ordinary vectors). For example, if $\mathbf{v} = \left[\begin{smallmatrix} 2 & 3 \end{smallmatrix}\right]$ and $\mathbf{w} = \left[\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right]$, then

$$\mathbf{vw} = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 23 \end{bmatrix}$$

and

$$\mathbf{v} \cdot \mathbf{w} = (2, 3) \cdot (4, 5) = 2 \cdot 4 + 3 \cdot 5 = 23.$$

Matrix addition is associative, which means that

$$(A + B) + C = A + (B + C).$$

Matrix addition is also commutative:

$$A + B = B + A.$$

Matrix multiplication is associative:

$$(AB)C = A(BC).$$

However, there are matrices A and B for which

$$AB \neq BA.$$

In other words, matrix multiplication is not commutative. For example, let $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]$. Then

$$AB = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \qquad \text{and} \qquad BA = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

The distributive properties hold:

$$A(B + C) = AB + AC \qquad \text{and} \qquad (A + B)C = AC + BC.$$

The $n \times n$ **identity matrix** is

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

(The symbol I is also used for identity matrices.) If A is $m \times n$, then $AI_n = A$. If B is $n \times p$, then $I_n B = B$.

The $m \times n$ **zero matrix** is the $m \times n$ matrix $\mathbf{0}$ with all entries equal to 0. If A is any matrix, then $A + \mathbf{0} = A$ and $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$ whenever the operations are defined.

An $m \times n$ matrix is **square** if $m = n$. A square matrix A is **upper triangular** if $A_{ij} = 0$ for $i > j$. A is therefore of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & A_{nn} \end{bmatrix}.$$

(All the entries below the main diagonal are zero.) A square matrix if **lower triangular** if all the entries above the main diagonal are zero.

If A is an $m \times n$ matrix, the **transpose** of A is the $n \times m$ matrix A^T for which $A^T_{ij} = A_{ji}$. A^T is obtained from A by interchanging rows and columns. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

An $n \times n$ matrix A is **invertible** (or **nonsingular**) if there is an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

For example, if $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 7 \end{smallmatrix}\right]$, then A is invertible and $A^{-1} = \left[\begin{smallmatrix} 7 & -2 \\ -3 & 1 \end{smallmatrix}\right]$. More generally, if $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right].$$

If A^{-1} exists, it is unique and is called the **inverse** of A. Non-invertible matrices are called **singular**. For example, the zero matrix is singular. If A and B are invertible, then

$$(A^{-1})^{-1} = A \qquad \text{and} \qquad (AB)^{-1} = B^{-1}A^{-1}.$$

To check if a matrix B is the inverse of A, one only needs to check one of the two conditions $AB = I$, $BA = I$. If one of the conditions is true, then the other is also true.

If A is invertible and m is a positive integer, then A^{-m} is defined to be

$$\underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{m}.$$

If A is invertible and m is any integer, then

$$(A^{-1})^m = (A^m)^{-1}.$$

If A is $n \times n$, then A^0 is defined to be I_n.

Problems for Section 2

1. If $\mathbf{x} = (1, 3, 4, 2)$ and $\mathbf{y} = (-4, 2, 5, -3)$, calculate $\mathbf{x} \cdot \mathbf{y}$.

2. Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}.$$

Calculate, if possible, $2A + 3B$, $A - B$, AB, CA, BC, CC^T and AB^T.

3. Show that if $A = \left[\begin{smallmatrix} 3 & 6 \\ 2 & 5 \end{smallmatrix}\right]$, then

$$A^{-1} = \begin{bmatrix} \frac{5}{3} & -2 \\ -\frac{2}{3} & 1 \end{bmatrix}$$

by showing that $AA^{-1} = I$.

4. If A is a 2×3 matrix with $A_{ij} = i^2 + 3j + 2$, write A as a rectangular array.

5. Show that I_n is invertible and $I_n^{-1} = I_n$.

6. Give an example of a non-zero 3×3 matrix which is not invertible.

7. Find two 2×3 matrices A and B so that $AB^T \neq BA^T$.

8. For each n, find an $n \times n$ matrix J_n so that $J_n \neq I_n$ and $J_n^2 = I_n$.

9. Let $A = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 5 \end{smallmatrix}\right]$. Find A^{-1}.

3 Linear Systems and Matrices

3.1 Matrix Form

Suppose we have a linear system of m equations in n variables $x_1, \ldots, x_n$, written as

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

We can write the system in the **matrix form** $AX = \mathbf{b}$, where A is the $m \times n$ **coefficient matrix**

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

For example, the system

$$\begin{aligned} x + 2y - z &= 3 \\ 3x - y + 4z &= 1 \end{aligned}$$

can be written as $AX = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

3.2 Reduced Row Echelon Form

As we saw above, every linear system of equations can be written in matrix form. For example, the system

$$\begin{aligned} x + 2y + 3z &= 4 \\ 3x + 4y + z &= 5 \\ 2x + y + 3z &= 6 \end{aligned}$$

can be written as

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

It is also useful to form the **augmented matrix**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 5 \\ 2 & 1 & 3 & 6 \end{bmatrix}.$$

Note that the fourth column consists of the numbers in the system on the right side of the equal signs.

If the augmented matrix has a particularly simple form, then the system is very easy to solve.

Definition. *A matrix is in* **reduced row echelon form (RREF)** *if*

1. The nonzero rows appear above the zero rows.

2. In any nonzero row, the first nonzero entry is a one (called the **leading one***).*

3. The leading one in a nonzero row appears to the left of the leading one in any lower row.

4. If a column contains a leading one, then all the other entries in that column are zero.

Example. Each of the following matrices is not in RREF.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example. Each of the following matrices is in RREF.

$$\begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix of any system of linear equations can be transformed into RREF by performing a series of operations on the rows of the matrix. The general plan is to first transform the entries in the lower left into zeros. The final step is to transform all the entries above the leading ones into zeros. The allowable operations are called **elementary row operations**. They are:

1. Divide a row by a nonzero number.
2. Subtract a multiple of a row from another row (or add a multiple of a row to another row).
3. Interchange two rows.

Performing any of these operations on an augmented matrix leads to a new system of equations which has the same set of solutions as the original system.

Example. Consider the system

$$\begin{aligned} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1. \end{aligned}$$

The corresponding augmented matrix is

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

Step 1. (Row 1)→ $\frac{1}{2}$ (Row 1) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

Step 2. (Row 2)→(Row 2) − 2 (Row 1) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 4 & 10 & -1 & 1 \end{bmatrix}$$

Step 3. (Row 3)→ (Row 3) − 4 (Row 1) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{bmatrix}$$

Step 4. (Row 2) → $-\frac{1}{3}$ (Row 2) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -6 & -9 & -3 \end{bmatrix}$$

Step 5. (Row 3)→(Row 3) + 6 (Row 2) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

Step 6. (Row 3)→ $-\frac{1}{3}$ (Row 3) gives

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 7. (Row 1) → (Row 1) − 4 (Row 2) gives

$$\begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 8. (Row 2)→(Row 2) − (Row 3) gives

$$\begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Step 9. (Row 1) → (Row 1) + 2 (Row 3) gives

$$\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This matrix is in RREF.

To put a matrix into RREF with your calculator, put the scratchpad in calculate mode. Then press MENU,7,5 and enter the matrix by using the key to the right of the 9 key.

3.3 Finding the Solutions

Once the augmented matrix of a linear system is put into RREF, it is easy to find all the solutions. A column of the matrix which contains a leading one is called a **leading column**. A variable which corresponds to a leading column is called a **leading variable**. The non-leading variables are called **free variables**. To find all solutions to the system corresponding to the RREF (and therefore to the original system), just solve the equations for the leading variables in terms of the free variables.

Example. Continuing with the previous example, the RREF is

$$\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

There are 3 leading variables and no free variables. The linear system corresponding to the RREF is $x_1 = 11$, $x_2 = -4$, $x_3 = 3$. The equations are already solved for the leading variables. The system has the one solution $(11, -4, 3)$.

Example. Suppose that the RREF of the augmented matrix of a linear system is

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding system is

$$\begin{aligned} x_1 + x_3 + x_4 &= 3 \\ x_2 + 2x_4 &= 1. \end{aligned}$$

The leading variables are x_1, x_2. The free variables are x_3, x_4. Solving for the leading variables gives

$$\begin{aligned} x_1 &= -x_3 - x_4 + 3 \\ x_2 &= -2x_4 + 1. \end{aligned} \tag{3}$$

If x_3 and x_4 are assigned values, then x_1 and x_2 are determined by (3) and we obtain a solution (x_1, x_2, x_3, x_4). The solution can also be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 + 3 \\ -2x_4 + 1 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

For example, if we let $x_3 = x_4 = 0$, then the corresponding solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

If $x_3 = 1$, $x_4 = 0$, we obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

If we let $x_3 = x_4 = 1$, we obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Note that since x_3 and x_4 can take on infinitely many values, there are infinitely many solutions.

Any linear system will have either one solution, no solutions or infinitely many solutions. If the RREF has a row of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

then there are no solutions since the equation corresponding to this row is $0 = 1$, which has no solutions. If there is no such row in the RREF, then there is either one or infinitely many solutions. If there are solutions and there is a free variable, then there are infinitely many solutions. If there are solutions and there are no free variables, then there is exactly one solution.

Example. Consider the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + 6y &= 2. \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}.$$

The RREF is

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The variable y is free, so there are infinitely many solutions. Written in vector form, the solutions are all of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Example. For the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + 6y &= 3, \end{aligned}$$

the augmented matrix is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 3 \end{bmatrix}$$

and the RREF is

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the system has no solutions.

Example. For the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + 5y &= 2, \end{aligned}$$

the augmented matrix is

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \end{bmatrix}$$

and the RREF is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

so the system has the one solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3.4 Elementary Row Operations and Inverses

Fact. If A is $n \times n$, then A is invertible $\iff$ $RREF(A) = I$.

Fact. If a series of elementary row operations transforms A into I, then the same series of elementary row operations transforms I into A^{-1}.

For example, let

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}.$$

The following series of elementary row operations transforms A into I.

1. Interchange Row 2 and Row 3
2. Interchange Row 1 and Row 2
3. (Row 1) $\to \frac{1}{2}$ (Row 1)
4. (Row 2) $\to \frac{1}{3}$ (Row 2)
5. (Row 3) $\to \frac{1}{4}$ (Row 3)

Performing the same series of operations on I transforms I into

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}.$$

3.5 Solving Systems with Inverses

If $AX = \mathbf{b}$ is a linear system with A invertible, then there is a unique solution $X = A^{-1}\mathbf{b}$. This is seen by noting that the following four statements are equivalent:

1. $AX = \mathbf{b}$
2. $A^{-1}AX = A^{-1}\mathbf{b}$
3. $IX = A^{-1}\mathbf{b}$ (since $A^{-1}A = I$)
4. $X = A^{-1}\mathbf{b}$ (since $IX = X$)

For example, the unique solution to the system

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Problems for Section 3

1. Find the RREF of each of the following matrices by hand. Check your answers with your calculator.

a. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 7 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & -1 & 0 & 5 \\ 1 & 2 & 5 & 5 \end{bmatrix}$

2. Write each of the following systems in matrix form, find the RREF of the augmented matrix by hand, check the RREF with your calculator and find all the solutions to the system. Write your solutions in vector form.

a. $\begin{aligned} x + 2y &= 5 \\ 2x - y &= 1 \end{aligned}$

b. $\begin{aligned} x + y + 5z &= 5 \\ x - y - z &= 1 \end{aligned}$

c. $\begin{aligned} 3x + 4y - z &= 8 \\ 6x + 8y - 2z &= 3 \end{aligned}$

d. $\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y &= 2 \\ x + 5y + 8z &= 10 \end{aligned}$

e. $\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y &= 2 \\ 5x + y - 3z &= 2 \end{aligned}$

f. $\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y &= 2 \\ 5x + y - 3z &= 3 \end{aligned}$

3. Does each of the following systems have no solutions, one solution, or infinitely many solutions?

a. $\begin{aligned} 2x - y &= 3 \\ 4x - 2y &= 5 \end{aligned}$

b. $\begin{aligned} 2x - y &= 3 \\ 4x - 2y &= 6 \end{aligned}$

c. $\begin{aligned} 2x + y &= 1 \\ 4x - 2y &= 3 \end{aligned}$

4. For each of the following matrices A, determine if A is invertible by finding the RREF. If A is invertible, find A^{-1} by using row operations.

a. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

5. Solve each of the following systems by finding the inverse of the coefficient matrix.

a. $x + 2y = 2$
$3x + 7y = 5$

b. $x + z = 1$
$y = 2$
$x + 2z = 3$

4 Determinants

4.1 Basic Concepts

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the **determinant** of A, denoted $|A|$ or $\det(A)$, is a real number. If $n = 1$, then $|[a]| = a$. If $n = 2$, then

$$|\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]| = ad - bc.$$

To calculate $|A|$ for an $n \times n$ matrix, let M_{ij} be the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and the j-th column of A. (M_{ij} is called a **minor**.) For example, let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}.$$

To calculate M_{11}, we delete row 1 and column 1 of A to get the 2×2 matrix $\left[\begin{smallmatrix} 2 & 5 \\ 0 & 1 \end{smallmatrix}\right]$. Then M_{11} is the determinant $|\left[\begin{smallmatrix} 2 & 5 \\ 0 & 1 \end{smallmatrix}\right]| = 2$. Similarly, to calculate M_{23}, we delete row 2 and column 3 of A to get $\left[\begin{smallmatrix} 1 & 3 \\ 3 & 0 \end{smallmatrix}\right]$. M_{23} is the determinant $|\left[\begin{smallmatrix} 1 & 3 \\ 3 & 0 \end{smallmatrix}\right]| = -9$. Let

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

(C_{ij} is called a **cofactor**.) Note that if $i + j$ is even,

$$(-1)^{i+j} = 1$$

and if $i + j$ is odd,

$$(-1)^{i+j} = -1.$$

For the matrix A above, for example, we have

$$C_{11} = (-1)^{1+1} M_{11} = 2$$

and

$$C_{23} = (-1)^{2+3} M_{23} = (-1)(-9) = 9.$$

To calculate $|A|$, pick i with $1 \leq i \leq n$ and consider the i-th row of A, which consists of $a_{i1}, a_{i2}, \ldots, a_{in}$. Then

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

This is known as "expanding along the i-th row". Alternatively, pick j with $1 \leq j \leq n$. Consider the j-th column of A. It consists of $a_{1j}, a_{2j}, \ldots, a_{nj}$. Then

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij}.$$

This is known as "expanding along the j-th column". For example, again let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix}.$$

If we expand along the first row (so that $i = 1$), then

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\ &= 1\left|\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}\right| - 3\left|\begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix}\right| + 2\left|\begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix}\right| \\ &= 1(2-0) - 3(4-15) + 2(0-6) = 23. \end{aligned}$$

If we expand along the third row (so that $j = 3$), then

$$\begin{aligned} |A| &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \\ &= a_{13}M_{13} - a_{23}M_{23} + a_{33}M_{33} \\ &= 2\left|\begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix}\right| - 5\left|\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}\right| + 1\left|\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\right| \\ &= 2(0-6) - 5(0-9) + 1(2-12) = 23. \end{aligned}$$

A good strategy is to expand along a row or column that consists mostly of zeros. For example, let

$$A = \begin{bmatrix} 3 & 0 & 2 & 4 \\ 1 & 2 & 1 & 3 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}.$$

The third row has only one non-zero entry, so we expand along that row.

$$\begin{aligned}
|A| &= \sum_{j=1}^{4} a_{3j} C_{3j} \\
&= a_{32} C_{32} \ \ (\text{since } a_{31} = a_{33} = a_{34} = 0) \\
&= 3(-1)^{2+3} M_{32} \\
&= -3 \left| \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \right| = 27.
\end{aligned}$$

4.2 Properties of Determinants

1. If $A \to B$ by switching 2 rows, then $|B| = -|A|$.

2. If $A \to B$ by multiplying a row or column of A by k, then $|B| = k|A|$.

3. If $A \to B$ by adding or subtracting a multiple of a row or column to another row or column, then $|B| = |A|$.

4. If one row is a multiple of another row, then $|A| = 0$. Similarly for columns. In particular, if A has a row of zeros or a column of zeros, then $|A| = 0$.

5. If A is an upper triangular or lower triangular matrix, then $|A|$ is the product of the diagonal entries, so that if

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & A_{nn} \end{bmatrix},$$

then

$$|A| = A_{11} A_{22} \cdots A_{nn}.$$

These properties provide another method for calculating determinants. Properties 1-3 spell out how the determinant changes when we apply an elementary row operation. Suppose we transform A into a triangular matrix B with a series of elementary row operations. Then it is easy to calculate $|B|$ by using Property 5. We can relate $|A|$ to $|B|$ by using Properties 1-3. For example, let

$$A = \begin{bmatrix} 5 & 10 & 20 \\ 3 & 6 & 5 \\ 2 & 3 & 1 \end{bmatrix}.$$

We perform the following series of elementary row operations:

Step 1. (Row 1)$\rightarrow \frac{1}{5}$(Row 1) gives

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 6 & 5 \\ 2 & 3 & 1 \end{bmatrix}.$$

This operation multiplies $|A|$ by $\frac{1}{5}$.

Step 2. (Row 2)$\rightarrow$ (Row 2) - 3(Row 1) gives

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -7 \\ 2 & 3 & 1 \end{bmatrix}.$$

This operation leaves the determinant unchanged.

Step 3. (Row 3)$\rightarrow$(Row 3) - 2(Row 1) gives

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -7 \\ 0 & -1 & -7 \end{bmatrix}.$$

This operation leaves the determinant unchanged.

Step 4. Interchanging (Row 2) and (Row 3) gives

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -7 \end{bmatrix}.$$

This operation multiplies the determinant by -1. Let B be the matrix obtained by Step 4. B is an upper triangular, so by Property 5,

$|B| = 1(-1)(-7) = 7$. Also the net effect of Steps 1-4 is to multiply $|A|$ by

$$(\frac{1}{5})(1)(1)(-1) = -\frac{1}{5}.$$

Therefore,

$$|B| = -\frac{1}{5}|A|,$$

so $|A| = -5|B| = (-5)(7) = -35$.

4.3 Determinants and Inverses

In case you want to check if a square matrix is invertible without trying to calculate the inverse, the following theorem is useful.

Theorem. *If A is $n \times n$, then $|A| \neq 0 \iff A^{-1}$ exists.*

For example, if $A = \left[\begin{smallmatrix} 3 & 2 \\ 5 & 4 \end{smallmatrix}\right]$, then A is invertible since $|A| = 2$. If

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

then $|B| = 0$, so B is not invertible

4.4 Cramer's Rule

Cramer's Rule is a method for solving a system of linear equations $AX = \mathbf{b}$ when A is a square matrix and $|A| \neq 0$. Write

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Let A_i be the $n \times n$ matrix obtained from A by replacing the i-th column of A with the vector $\mathbf{b}$. Then

$$x_i = \frac{|A_i|}{|A|}.$$

For example, consider the linear system

$$\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}.$$

We have

$$A = \left[\begin{smallmatrix} 2 & 4 \\ 3 & 7 \end{smallmatrix}\right],\ A_1 = \left[\begin{smallmatrix} 5 & 4 \\ 9 & 7 \end{smallmatrix}\right] \text{ and } A_2 = \left[\begin{smallmatrix} 2 & 5 \\ 3 & 9 \end{smallmatrix}\right].$$

By Cramer's Rule, the solution is

$$x = \frac{|\left[\begin{smallmatrix} 5 & 4 \\ 9 & 7 \end{smallmatrix}\right]|}{|\left[\begin{smallmatrix} 2 & 4 \\ 3 & 7 \end{smallmatrix}\right]|} = -\frac{1}{2},\ y = \frac{|\left[\begin{smallmatrix} 2 & 5 \\ 3 & 9 \end{smallmatrix}\right]|}{|\left[\begin{smallmatrix} 2 & 4 \\ 3 & 7 \end{smallmatrix}\right]|} = \frac{3}{2}.$$

Problems for Section 4

1. Let $A = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 5 \end{smallmatrix}\right]$. Calculate $|A|$.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

a. Find M_{23}.

b. Find C_{23}.

c. Calculate $|A|$ by expanding along the first row.

d. Calculate $|A|$ by expanding along the second column.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & 7 & 8 \\ 2 & 0 & 1 & 3 \\ -1 & 0 & 0 & 2 \end{bmatrix}.$$

Calculate $|A|$ by using row operations to reduce A to a triangular matrix.

4. Let

$$A = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 2 & 3 & 1 & 2 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Calculate $|A|$ by expanding along the second column.

5. Use the Theorem in Section 4.3 to determine if the following matrices are invertible.

a. $\begin{bmatrix} 6 & 10 \\ 4 & 6 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & 8 & 2 \end{bmatrix}$

6. Use Cramer's Rule to solve the system

$$\begin{aligned} x + 2y &= 3 \\ 2x + 7y &= 5. \end{aligned}$$

5 Applications

We present several applications to economic models of the techniques for solving linear systems.

5.1 Market Model

By eliminating the quantity variables, the linear market model for two commodities can be written as

$$\begin{aligned} aP_1 + bP_2 &= u \\ cP_1 + dP_2 &= v, \end{aligned}$$

where P_1 and P_2 are the two prices. If $\left|\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right| \neq 0$, then this system can be solved by using Cramer's Rule. The equilibrium prices are

$$P_1 = \frac{\left|\begin{bmatrix} u & b \\ v & d \end{bmatrix}\right|}{\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|}, \ P_2 = \frac{\left|\begin{bmatrix} a & u \\ c & v \end{bmatrix}\right|}{\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|}.$$

5.2 National Income Model

The national income model is given by

$$\begin{aligned} Y &= C + I_0 + G_0 \\ C &= a + bY, \end{aligned}$$

where Y is national income, C is planned consumption expenditures, I_0 is investment and G_0 is government expenditure. The first equation says that national income equals total expenditure by consumers, business and government. The second equation says that expenditures by consumers (consumption) equals some baseline amount plus a fraction of the national income (so $0 < b < 1$). You may think of b as depending on consumers' mood. To solve this system for Y and C, we rewrite it as

$$\begin{aligned} Y - C &= I_0 + G_0 \\ -bY + C &= a. \end{aligned}$$

The coefficient matrix is $A = \left[\begin{smallmatrix} 1 & -1 \\ -b & 1 \end{smallmatrix}\right]$. Solving using Cramer's Rule, we get

$$Y = \frac{\left|\begin{smallmatrix} I_0+G_0 & -1 \\ a & 1 \end{smallmatrix}\right|}{|A|} = \frac{I_0 + G_0 + a}{1-b}$$

and

$$C = \frac{\left|\begin{smallmatrix} 1 & I_0+G_0 \\ -b & a \end{smallmatrix}\right|}{|A|} = \frac{a + b(I_0 + G_0)}{1-b}.$$

5.3 Leontief Input-Output Models

Suppose we have n factories and consumers. There is consumer demand for the products of the factories. In addition, each factory needs products from their own factory and the other factories in order to produce their product. We would like to find the output level for each factory so that all the demands are met with no product left over. Let d_i be the consumer demand for product i. Let x_i be the output of factory i. We measure the demand and output in terms of units (which could be dollars). Let a_{ij} be the number of units that factory i sends to factory j for each unit that factory j produces. Let A be the $n \times n$ matrix $[a_{ij}]$. When $n = 2$, we have the following system of equations:

$$\begin{aligned} x_1 &= a_{11}x_1 + a_{12}x_2 + d_1 \\ x_2 &= a_{21}x_1 + a_{22}x_2 + d_2. \end{aligned}$$

This can be written as

$$\begin{aligned} (1 - a_{11})x_1 - a_{12}x_2 &= d_1 \\ -a_{21}x_1 + (1 - a_{22})x_2 &= d_2. \end{aligned}$$

Writing this in matrix form gives

$$\begin{bmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

which can be written as

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

or

$$(I - A)X = D.$$

The matrix $I - A$ is called the **Leontief matrix**. If $(I - A)$ is invertible, then the output vector is

$$X = (I - A)^{-1}D.$$

This formula is also valid for the general case of n factories.

Example. Suppose we have 2 factories. Factory 1 (a power plant) produces electricity and Factory 2 (a municipal water facility) produces water. To have output from the two factories measured in common units, we measure both outputs in dollars. For each unit of electricity produced, Factory 1 must use .2 units of electricity and .4 units of water. For each unit of water produced, Factory 2 must use .3 units of electricity and .1 units of water. Also, consumer demand is 10 units of electricity and 30 units of water. We have

$$A = \begin{bmatrix} .2 & .3 \\ .4 & .1 \end{bmatrix}.$$

The output vector is

$$X = (I - A)^{-1}D = \begin{bmatrix} .8 & -.3 \\ -.4 & .9 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 30 \end{bmatrix} = \begin{bmatrix} 30 \\ 46.7 \end{bmatrix}.$$

We assume that each factory spends \$1 to produce \$1 worth of output. Since

$$\sum_{i=1}^{n} a_{ij}$$

is the amount per unit of output that factory j spends on the products it receives from the n factories (including itself), there is

$$1 - \sum_{i=1}^{n} a_{ij}$$

per unit left over. This amount is spent on labor. Let a_{0j} denote this amount. Factory j therefore spends $a_{0j}x_j$ on labor. The total labor cost for all n factories is

$$\sum_{j=1}^{n} a_{0j}x_j. \tag{4}$$

It can be shown that the total labor cost (4) also equals

$$\sum_{i=1}^{n} d_i, \tag{5}$$

so it is possible to calculate the total labor cost without solving for the outputs x_j.

Example. Using the setup from the previous example, we see that the total labor cost is $d_1 + d_2 = 10 + 30 = 40$. We have

$$a_{01} = 1 - a_{11} - a_{21} = 1 - .2 - .4 = .4$$

and

$$a_{02} = 1 - a_{12} - a_{22} = 1 - .3 - .1 = .6.$$

Since $x_1 = 30$ and $x_2 = 46.7$, we see that Factory 1 spends $(.4)(30) = 12$ on labor and Factory 2 spends $(.6)(46.7) = 28$ on labor. Note that the total spent by both factories is therefore $12 + 28 = 40$, which agrees with the alternate calculation for total labor cost of $10 + 30 = 40$.

Problems for Section 5

1. Consider the two commodity market model given by

$$
\begin{aligned}
D_1 &= S_1 \\
D_1 &= 10 - 2P_1 + P_2 \\
S_1 &= -2 + 3P_1 \\
D_2 &= S_2 \\
D_2 &= 15 + P_1 - P_2 \\
S_2 &= -1 + 2P_2.
\end{aligned}
$$

Find the equilibrium prices by using Cramer's Rule.

2. Consider the three commodity market model given by

$$
\begin{aligned}
D_1 &= S_1 \\
D_1 &= 20 - P_1 + 2P_2 + P_3 \\
S_1 &= -2 + 13P_1 \\
D_2 &= S_2 \\
D_2 &= 25 + 2P_1 - P_2 + 2P_3 \\
S_2 &= -1 + 12P_2 \\
D_3 &= S_3 \\
D_3 &= 30 + 3P_1 + 2P_2 - P_3 \\
S_3 &= -3 + 14P_3
\end{aligned}
$$

Find the equilibrium prices.

3. Suppose that in a national income model as above, we have $I_0 = 8$, $G_0 = 5$, $a = 4$ and $b = \frac{1}{3}$. Use Cramer's Rule to find Y and C.

4. Suppose we have a facility (E) that produces electricity and a facility (C) that sells chemicals. For each unit of electricity that E produces, it uses .4 units of electricity and .5 units of chemicals. For each unit of chemicals that C produces, it uses .1 units of electricity and .5 units of chemicals. The consumer demand is 20 units of electricity and 10 units of chemicals.

a. Find the Leontief matrix $I - A$ for the input-output model.

b. Find the output levels for each facility so that the demands of both the facilities and the consumers are satisfied.

c. Calculate the total labor cost by using equation (4).

d. Calculate the total labor cost by using equation (5).

e. How much does E pay C?

f. How much does C pay E?

g. How much does E pay for labor?

h. How much does C pay for labor?

5. We have 3 factories. Factory 1 produces plastic, factory 2 produces rubber and factory 3 produces metal. For each unit that factory 1 produces, it uses .1 units of plastic, .2 units of rubber and .2 units of metal. For each unit that factory 2 produces, it uses .2 units of rubber, .3 units of plastic and .1 units of metal. For each unit that factory 3 produces, it uses .3 units of metal, .2 units of plastic and .4 units of rubber. The consumer demand is 70 units of plastic, 50 units of rubber and 30 units of metal.

a. Find the output levels for each factory so that the demands of both the factories and the consumers are satisfied.

b. Find the total labor cost.

c. For $1 \leq i \leq 3$ and $1 \leq j \leq 3$, how much does factory i pay factory j?

d. How much does each factory pay for labor?

6 Discrete Dynamical Systems

Suppose that A_t is the amount of money we have in a bank account at time t. Our initial deposit is therefore A_0. Suppose that the bank pays interest annually at the rate r. This means that after one year, the bank pays us interest of rA_0, so we have in our account

$$A_1 = A_0 + rA_0 = (1 + r)A_0.$$

After one more year, we have

$$A_2 = A_1 + rA_1 = (1 + r)A_1.$$

In general, we have

$$A_{n+1} = A_n + rA_n = (1 + r)A_n.$$

We would like to know how much we will have in our account after n years, where $n = 1, 2, 3, ...$ This amount will depend on the initial deposit A_0 and the interest rate r. This problem is easily solved by writing

$$\begin{aligned}
A_1 &= (1 + r)A_0 \\
A_2 &= (1 + r)A_1 = (1 + r)(1 + r)A_0 = (1 + r)^2 A_0 \\
A_3 &= (1 + r)A_2 = (1 + r)(1 + r)^2 A_0 = (1 + r)^3 A_0 \\
&\vdots \\
A_n &= (1 + r)^n A_0.
\end{aligned}$$

For example, if $A_0 = 100$ and $r = .05$, then after 10 years, we have

$$A_{10} = (1.05)^{10} 100 \approx 162.89.$$

Suppose, more generally, that we measure a quantity at times $n = 0, 1, 2, 3...$, and let A_n be the quantity at time n. A **discrete dynamical system** is an equation which describes a relationship between the quantity at a point in time and the quantity at earlier points in time. We will also call these **dynamical systems** and use the abbreviation **DS**. If the quantity depends only on the quantity at the previous point in time, we will call this a **first order** dynamical system and write

$$A_{n+1} = f(A_n),$$

where $f(x)$ is a function of one variable. We can also think of a first order DS as a sequence of numbers for which there is a rule that relates each number in the sequence to the previous number in the sequence. For example, the DS

$$A_{n+1} = (1 + r)A_n,\ n \geq 0,$$

corresponds to the sequence

$$A_0, (1 + r)A_0, (1 + r)^2 A_0, \ldots.$$

Another DS is given by the equation

$$A_{n+2} = A_{n+1} + A_n,\ n \geq 0.$$

If we specify that $A_0 = A_1 = 1$, then we obtain the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

This is the well-known **Fibonacci** sequence. It is not a first order DS since each member of the sequence depends on the previous two members. We can write

$$A_{n+2} = g(A_n, A_{n+1}),$$

where $g(x, y) = x + y$.

For the rest of this section we will consider only first order DS $A_{n+1} = f(A_n)$. If $f(x)$ is a function of the form $f(x) = sx + b$, we will say the DS is **linear**. Otherwise, the DS is **nonlinear**. For example, the DS $A_{n+1} = 3A_n$ is linear, since $f(x) = 3x$, but $A_{n+1} = 3A_n - A_n^2$ is nonlinear, since $f(x) = 3x - x^2$.

Given a DS $A_{n+1} = f(A_n)$, we would like to find a sequence of numbers $A_0, A_1, \ldots$ which satisfies the DS for all $n \geq 0$. We will refer to such a sequence as a **solution** to the DS. For example, a solution to the DS $A_{n+1} = 3A_n$ is the sequence $A_n = c\,3^n$, where c is any real number, since

$$A_{n+1} = c\,3^{n+1} = 3(c\,3^n) = 3A_n.$$

We therefore have infinitely many solutions to the DS. In fact, any solution to this DS is of the form $A_n = c\,3^n$ for some real number c. We say that $A_n = c\,3^n$ is the **general solution** to the DS. If we also specify the value of

A_0, then we say that we are given an **initial condition** (IC). For example, suppose that $A_0 = 5$. Then if $A_n = c\,3^n$ is a solution to the DS which satisfies the IC, we must have

$$5 = A_0 = c\,3^0 = c.$$

Therefore, there is exactly one solution to the DS which satisfies the IC. It is $A_n = 5 \cdot 3^n$. A solution to a DS which also satisfies an IC is called a **particular solution**.

Suppose that a DS $A_{n+1} = f(A_n)$ has the solution

$$A_n = c,\; n = 0, 1, 2, \ldots$$

for some real number c. The quantity A_n is unchanging as n changes. The system involving the quantities A_n is in an equilibrium state and the number c is called an **equilibrium value** or a **fixed point** of the DS. Also, $A_n = c$ is called a **constant solution** to the DS. We then have $A_{n+1} = A_n = c$ for all n, so $A_{n+1} = f(A_n)$ implies that $c = f(c)$. Conversely, if $c = f(c)$, then $A_n = c$ gives a constant solution. Therefore, to find all the fixed points of the DS, we only need to solve the equation $c = f(c)$.

Example 1. If $A_{n+1} = 4A_n - 2$, we have $f(x) = 4x - 2$, so $c = f(c)$ gives $c = 4c - 2$ and the only fixed point is $c = \frac{2}{3}$. If $A_0 = \frac{2}{3}$, then

$$A_1 = 4\left(\frac{2}{3}\right) - 2 = \frac{2}{3}, \qquad A_2 = 4\left(\frac{2}{3}\right) - 2 = \frac{2}{3}, ...$$

and so $A_n = \frac{2}{3}$ for all $n = 1, 2, 3...$

Example 2. If $A_{n+1} = sA_n + b$ is the general first order linear DS, then $f(x) = sx + b$ and the fixed points are the solutions to $c = sc + b$. If $s \neq 1$, the only solution is

$$c = \frac{b}{1 - s}.$$

If $s = 1$ and $b \neq 0$, there are no fixed points. If $s = 1$ and $b = 0$, then the DS is $A_{n+1} = A_n$ and any number c is a fixed point.

The solution to a DS can sometimes be found by writing out the first few cases of $A_{n+1} = f(A_n)$, simplifying so that each A_{n+1} is expressed in terms of A_0 and noticing a pattern. For example,we saw that if $A_{n+1} = (1+r)A_n$, then $A_n = (1+r)^n A_0$. Similary, if $A_{n+1} = A_n + b$, then we have

$$\begin{aligned} A_1 &= A_0 + b \\ A_2 &= A_1 + b = A_0 + 2b \\ &\vdots \\ A_n &= A_0 + nb \end{aligned}$$

and we see that the solution is $A_n = A_0 + nb$.

Problems for Section 6.

1. We deposit \$1000 in the the bank. The interest rate is 6%, compounded annually. How much do we have after 15 years?

2. We deposit \$1000. The annual rate is 6%, compounded monthly. How much do we have after 15 years?

3. Suppose the general solution to a DS is $A_n = c7^n$. The initial condition is $A_0 = 3$. Find the particular solution that satisfies the IC.

4. The DS is $A_{n+1} = A_n^2 - 5A_n + 6$. Find the fixed points.

5. The DS is $A_{n+1} = 3A_n - 2$. Find a formula for A_n in terms of A_0 by finding A_1, A_2 and A_3 and guessing the pattern.

7 Interest Rates

We saw that if r is the annual interest rate and interest is compounded annually, then the amount we have in our account after n years is

$$A_n = A_0(1+r)^n,$$

where A_0 is the initial amount. Now suppose that r is the annual rate but that we compound monthly instead of annually. At the end of each month, we earn interest at the rate of $\frac{r}{12}$ on the amount we had in the account during that month. The amount we have after n months is determined by the DS

$$A_{n+1} = A_n\left(1+\frac{r}{12}\right),$$

and the solution to the DS is

$$A_n = A_0\left(1+\frac{r}{12}\right)^n.$$

More generally, suppose we split the year up into k equal pieces (for example, $k = 1, 12, 52, 365$). The pieces are also called time units or periods. A_n represents the amount we have after n periods, each period being $\frac{1}{k}$ years. With an annual rate of r, we compound at the end of each of the k periods, earning interest at the rate of $\frac{r}{k}$ on the amount we had during that period. The DS in this case is

$$A_{n+1} = A_n\left(1+\frac{r}{k}\right),$$

and the solution is

$$A_n = A_0\left(1+\frac{r}{k}\right)^n.$$

If we want the amount in the account after t years, we convert this to kt periods and the amount is

$$A_{kt} = A_0\left(1+\frac{r}{k}\right)^{kt}.$$

Example. Suppose $A_0 = 100$ and the annual rate is $r = .05$. The amounts A_{10k} we have in the account after 10 years for various values of k are:

k	amount
1=annually	162.889
12=monthly	164.701
52=weekly	164.833
365=daily	164.866
(365)(24)=hourly	164.872
∞=continuously	164.872

Note that the last two entries are equal up to three decimal places. Each line in the table except for the last is computed using

$$A_{10k} = 100\left(1 + \frac{.05}{k}\right)^{10k}.$$

The last entry is computed using the formula (which you may have learned in calculus)

$$A = A_0 e^{rt},$$

with $A_0 = 100$, $r = .05$ and $t = 10$. In general, as the number of compounding periods k in a year approaches infinity, we have

$$A_0\left(1 + \frac{r}{k}\right)^{kt} \to A_0 e^{rt}.$$

Problems for Section 7.

1. We make an initial deposit of A_0 in our account. The annual interest rate is r, compounded 52 times per year.

a. Find the dynamical system (DS) which models this situation.

b. Find the solution to the DS in **a.**

c. Find the fixed points of the DS if $r \neq 0$.

d. Assume that $A_0 = 100$ and $r = .08$. Find the amount in the account after 5 years.

2. Suppose the annual rate is .05, compounded monthly. How much should we deposit initially so that we have 10,000 in 40 years?

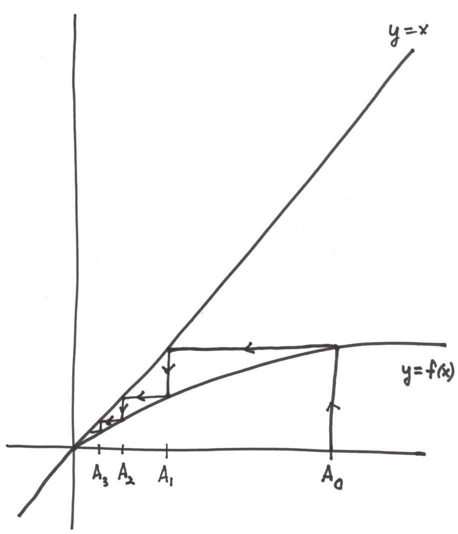
3. We deposit 1000 initially. Find the smallest annual rate which will let us accumulate at least 2000 over 10 years if interest is compounded

a. annually **b.** daily

8 Cobwebs

Cobwebs are a graphical method for understanding discrete dynamical systems $A_{n+1} = f(A_n)$. They are constructed as follows:

1. Draw the graph of $y = f(x)$,
2. Draw the line $y = x$.
3. Pick an initial point A_0 on the x-axis.
4. Connect $(A_0, 0)$ to (A_0, A_1) with a vertical line. Note that $A_1 = f(A_0)$, so (A_0, A_1) is on the graph of $y = f(x)$.
5. Connect (A_0, A_1) to (A_1, A_1) with a horizontal line.
6. Connect (A_1, A_1) to (A_1, A_2) with a vertical line. Note that $A_2 = f(A_1)$, so (A_1, A_2) is on the graph of $y = f(x)$.
7. Continue in this way.

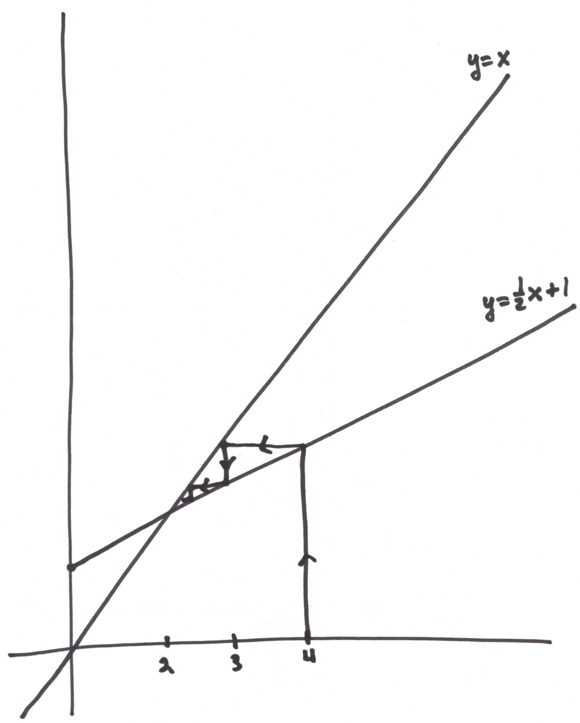


Example. Suppose our DS is $A_{n+1} = \frac{1}{2}A_n + 1$. Choose $A_0 = 4$. Then the DS evolves as:

$$4 \to 3 \to 2.5 \to 2.25 \to 2.125 \to \cdots .$$

We connect the following sequence of points to form a cobweb:

$$(4,0) \to (4,3) \to (3,3) \to (3,2.5) \to (2.5,2.5) \to (2.5,2.25) \to (2.25,2.25) \to \cdots$$



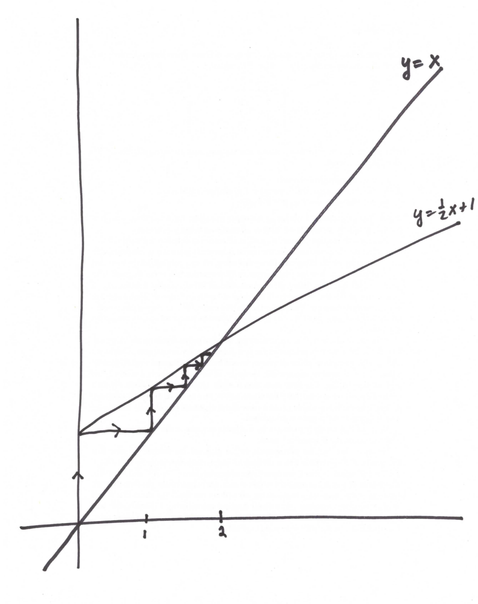
If $A_0 = 0$, then the DS evolves as:

$$0 \to 1 \to 1.5 \to 1.75 \to 1.875 \to \cdots .$$

We connect the sequence

$$(0,0) \to (0,1) \to (1,1) \to (1,1.5) \to (1.5,1.5) \to (1.5,1.75) \to \cdots$$

The pictures indicate that for any starting point A_0, $A_n \to 2$ as $n \to \infty$. It



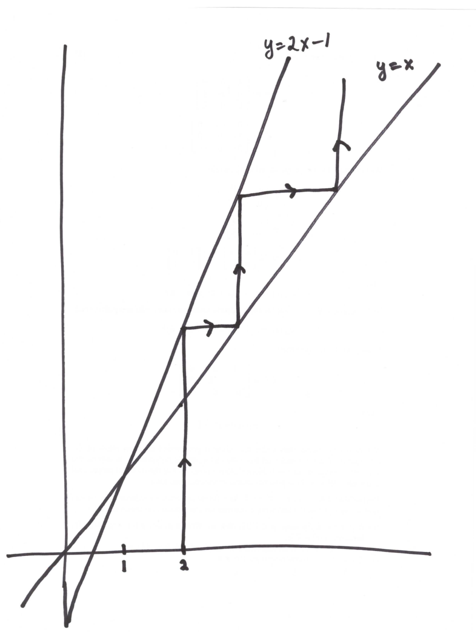
appears that successive points are being attracted to the point $(2,2)$. Note that 2 is the only fixed point. The point 2 is a an example of an **attracting** fixed point.

Example. Suppose our DS is $A_{n+1} = 2A_n - 1$. Choose $A_0 = 2$. Then the DS evolves as:

$$2 \to 3 \to 5 \to 9 \to 17 \to \cdots .$$

We connect the following sequence of points to form a cobweb:

$$(2,0) \to (2,3) \to (3,3) \to (3,5) \to .5,5) \to (5,9) \to (9,9) \to \cdots 2$$



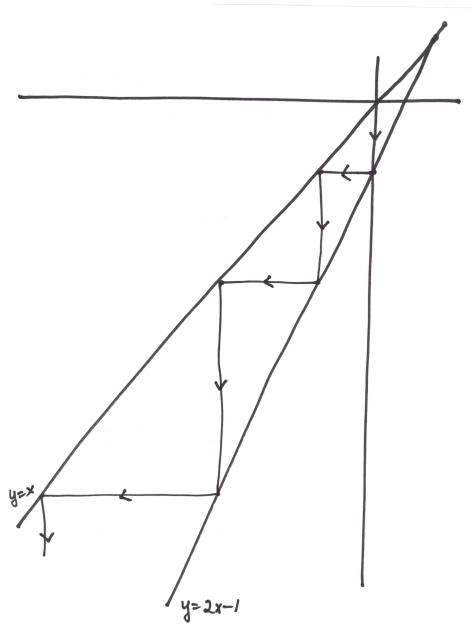
If $A_0 = 0$, then the DS evolves as:

$$0 \to -1 \to -3 \to -5 \to -7 \to \cdots .$$

We connect the sequence

$$(0,0) \to (0,-1) \to (-1,-1) \to (-1,-3) \to (-3,-3) \to (-3,-7) \to \cdots$$

The picture indicates that for any starting point A_0, $|A_n| \to \infty$ as $n \to \infty$.



It appears that successive points are being repelled from the point $(1,1)$. Note that 1 is the only fixed point. The point 1 is an example of a **repelling** fixed point.

Roughly speaking, a fixed point c for a DS is **attracting** if whenever A_0 is sufficiently close to c, then $A_n \to c$ as $n \to \infty$. A fixed point c is **repelling** if no matter how close A_0 is to c, A_n is eventually far away from c at infinitely many times. There are DS which have a fixed point which is neither attracting nor repelling.

Problems for Section 8.

For each of the following DS $A_{n+1} = f(A_n)$, find the fixed points and use cobwebs to determine whether each fixed point is attracting, repelling or neither.

1. $f(x) = \frac{1}{3}x + 2$

2. $f(x) = 3x - 2$

3. $f(x) = -x + 2$

9 First Order Linear Dynamical Systems

9.1 General and Particular Solutions

Consider a first order linear DS

$$A_{n+1} = sA_n + b. \tag{6}$$

Theorem. *If $s \neq 1$, the general solution to (6) is*

$$A_n = ks^n + \frac{b}{1-s}. \tag{7}$$

If $s = 1$, the general solution to (6) is

$$A_n = k + nb. \tag{8}$$

In other words, (7) and (8) are solutions to (6) for any k and any solution to (6) is one of these forms. If $s \neq 1$ and A_0 is specified with an IC, then we must have

$$A_0 = ks^0 + \frac{b}{1-s} = k + \frac{b}{1-s},$$

so

$$k = A_0 - \frac{b}{1-s}$$

and therefore

$$A_n = \left(A_0 - \frac{b}{1-s}\right) s^n + \frac{b}{1-s} \tag{9}$$

is the particular solution which satisfies the IC. Similarly, if $s = 1$ and A_0 is specified, then $k = A_0$ and the particular solution is

$$A_n = A_0 + nb.$$

Example. We have a bank account earning 5% interest, compounded annually. We deposit 500 initially and also deposit 100 at the end of each year. How much do we have after 10 years? To solve this, let A_n be the amount we have after n years. Then we have the DS

$$A_{n+1} = 1.05A_n + 100.$$

We also have the IC $A_0 = 500$. By using equation (9) with $A_0 = 500$, $s = 1.05$ and $b = 100$, the particular solution is

$$A_n = \left(500 - \frac{100}{(-.05)}\right)(1.05)^n + \frac{100}{(-.05)} = 2500(1.05)^n - 2000.$$

We therefore have $A_{10} = 2072.24$.

Example. Suppose now that the interest in our account is compounded monthly. If the annual rate is .12, the monthly rate is $\frac{.12}{12} = .01$. Assume we deposit 50 initially and deposit 10 at the end of each month. The DS for this problem is

$$A_{n+1} = (1.01)\,A_n + 10.$$

This is a linear DS, with $s = 1.01$ and $b = 10$. The IC is $A_0 = 50$. Using equation (9), the solution is

$$A_n = 1050(1.01)^n - 1000.$$

After 10 years (120 months), we have

$$A_{120} = 1050(1.01)^{120} - 1000 = 2465.41.$$

Compare this with how much we would have with annual compounding. In this case, the DS is

$$A_{n+1} = (1.12)A_n + 120$$

and the solution is, using (9),

$$A_n = \left(50 - \frac{120}{1 - 1.12}\right)(1.12)^n + \frac{120}{1 - 1.12} = 1050(1.12)^n - 1000,$$

which gives $A_{10} = 2261.14$, showing that monthly compounding is substantially better.

Example. Suppose we win the lottery. We can take 500,000 now or 50,000 in 20 annual payments, with the first payment given now. Which payment option leaves us with the most money at the time we get the last of the 20 payments? Assume that whenever we get a payment, we put it in an account earning interest at a annual rate of r, compounded annually. Let

A_n be the amount we have in the account after n years if we choose the first option. Let B_n be the amount for the second option. Then we have

$$A_{n+1} = (1+r)A_n$$

with $A_0 = 500,000$. Therefore, $A_n = 500,000(1+r)^n$. Also,

$$B_{n+1} = (1+r)B_n + B_0$$

with $B_0 = 50,000$. Therefore,

$$B_n = \left(B_0 - \frac{B_0}{1-(1+r)}\right)(1+r)^n + \frac{B_0}{1-(1+r)} = \frac{B_0}{r}((1+r)^{n+1} - 1),$$

We get the final payment after 19 years have elapsed, so we should compare A_{19} and B_{19}. Suppose $r = .05$. Then $A_{19} \approx 1,263,480$ and $B_{19} \approx 1,653,300$, so the second option is better. If $r = .1$, then $A_{19} \approx 3,057,900$ and $B_{19} \approx 2,863,750$, so the first option is better.

9.2 Fixed Points

We saw in Example 2 of Section 6 that a linear system $A_{n+1} = sA_n + b$ has the single fixed point

$$\frac{b}{1-s} \tag{10}$$

if $s \neq 1$. If $s = 1$ and $b \neq 0$, there are no fixed points. If $s = 1$ and $b = 0$, then $f(x) = x$, so all points are fixed points.

If the DS is linear and $f(x) = sx + b$ with $s \neq 1$, equation (6) that the general solution is

$$A_n = ks^n + \frac{b}{1-s}.$$

If $|s| < 1$, then as $n \to \infty$, $s^n \to 0$, so

$$A_n \to \frac{b}{1-s},$$

showing that the fixed point $\frac{b}{1-s}$ is attracting, since no matter what the intial value A_0 is, the terms A_n always approach $\frac{b}{1-s}$. If $|s| > 1$, then as

$n \to \infty$, $|s^n| \to \infty$, so the sequence A_n has no finite limit. In this case, the fixed point is repelling. If $s = -1$, then the DS is

$$A_{n+1} = -A_n + b,$$

so

$$A_0 \to -A_0 + b \to A_0 \to -A_0 + b \to A_0 \to \cdots$$

showing that $A_0 = A_2 = A_4 = \ldots$ and $A_1 = A_3 = A_5 = \ldots$. The sequence A_n neither approaches nor gets far away from the fixed point, so the fixed point is neither attracting nor repelling.

9.3 Discrete Market Models

In a **discrete market model**, prices, supply and demand are measured only at the times $t = 0, 1, 2, \ldots$. At time t, P_t is the price, D_t is the demand and S_t is the supply. Suppose that the supply at time t is determined by the price at time $t-1$ and that the demand at time t is determined by the price at time t. Such a situation could occur, for example, with a farmer planting crops. The supply of the crop at time t is determined by how much crop he plants at time $t-1$ (assuming that it takes one time period for the crop to grow, be harvested and taken to market), and the amount he plants is determined by the market price at time $t-1$. We assume supply and demand are **linear** functions of price. The equations of the market model are therefore

$$\begin{aligned} D_t &= S_t \\ D_t &= a - bP_t \\ S_t &= -c + dP_{t-1}. \end{aligned}$$

We assume that a, b, c, d are all positive. Substituting the second and third equations into the first equation, we obtain

$$a - bP_t = -c + dP_{t-1}.$$

Changing t to $t+1$ and rearranging gives

$$P_{t+1} = \left(-\frac{d}{b}\right) P_t + \frac{a+c}{b}.$$

This is a 1st order linear DS. Using (7). the general solution is

$$P_t = k\left(-\frac{d}{b}\right)^t + \frac{a+c}{b+d},$$

where k is an arbitrary constant. Using (10), the single fixed point is

$$\overline{P} = \frac{a+c}{b+d}.$$

and we write the general solution as

$$P_t = k\left(-\frac{d}{b}\right)^t + \overline{P}.$$

Letting $t = 0$ gives $P_0 = k + \overline{P}$, so that $k = P_0 - \overline{P}$ and

$$P_t = (P_0 - \overline{P})\left(-\frac{d}{b}\right)^t + \overline{P}.$$

Notice that $P_t \to \overline{P}$ when

$$\left(-\frac{d}{b}\right)^t \to 0$$

and this happens only when $\left|\frac{d}{b}\right| < 1$. Note that if $P_0 = \overline{P}$, then $P_t = \overline{P}$ for all t. This is to be expected since $\overline{P}$ is a fixed point.

Problems for Section 9.

1. Find the general solution to the DS $A_{n+1} = 3A_n - 2$. Find the particular solution which satisfies $A_0 = 3$.

2. Find the general solution to the DS $A_{n+1} = A_n + 4$. Find the particular solution which satisfies $A_0 = 1$.

3. We make an initial deposit of 900 in our bank account. We make additional deposits of 50 at the end of each month for the next 2 years (24 months). The annual interest rate is .048, compounded monthly.

a. Let A_n be the amount in the account after n months. State the dynamical system whose solution is the sequence A_n.

b. Find the general solution to the dynamical system.

c. Find the particular solution that satisfies the initial condition.

d. Use the particular solution to find the amount in the account after 2 years.

4. We deposit 200 in our account initially and then deposit 60 at the end of each year. With an annual rate of .03, compounded annually, how much do we have after 5 years?

5. We deposit 1000 initially and we withdraw 10 at the end of each month. The annual rate is .06, compounded monthly (so the monthly rate is .005). Find n so that A_n is positive and A_{n+1} is negative. In other words, how long does the money last?

6. We win a small lottery prize. We can take 1000 now or take 30 annual payments of 45 each, the first being given now. The annual interest rate is .02, compounded annually. Which payment option leaves us with more money at the time of the last annual payment?

7. Consider the discrete market model

$$\begin{aligned} D_t &= S_t \\ D_t &= 22 - 3P_t \\ S_t &= -2 + P_{t-1}, \end{aligned}$$

with $P_0 = 8$.

a. Find a, b, c, d.

b. Find the equilibrium price $\overline{P}$.

c. Find the sequence P_t which satisfies the model.

d. As $t \to \infty$, is P_t converging? If so, to which number?

8. For the discrete market model

$$\begin{aligned} D_t &= S_t \\ D_t &= 7 - P_t \\ S_t &= -5 + P_{t-1}, \end{aligned}$$

with $P_0 = 7$, answer the questions **a-d** from Problem 7.

10 Second Order Dynamical Systems

A **second order** DS is a DS of the form

$$A_{n+2} = f(A_{n+1}, A_n), n = 0, 1, 2, \ldots$$

For example,

$$A_2 = f(A_1, A_0) \text{ and } A_3 = f(A_2, A_1).$$

In order to determine $A_2, A_3, \ldots$, we must be given A_0 and A_1. These two values are called the **initial conditions** of the DS.

One class of second order DS are those of the form

$$A_{n+2} = aA_{n+1} + bA_n + c. \tag{11}$$

These are the **linear** second order DS, and they always have solutions.

Theorem. *If we are given initial conditions A_0 and A_1, the DS (11) has a unique solution.*

To find the general solution to DS of this form, consider the **characteristic equation**

$$x^2 = ax + b. \tag{12}$$

Let r and s be the roots of (12).

Theorem. *If $a + b \neq 1$, the general solution to (11) is*

$$A_n = \begin{cases} c_1 r^n + c_2 s^n + \frac{c}{1-a-b}, & \text{if } r \neq s; \\ \\ (c_1 + c_2 n) r^n + \frac{c}{1-a-b}, & \text{if } r = s. \end{cases}$$

If $a + b = 1$, the general solution to (11) is

$$A_n = \begin{cases} c_1 (a-1)^n + c_2 + \left(\frac{c}{2-a}\right) n, & \text{if } a + b = 1, a \neq 2; \\ \\ c_1 + c_2 n + \left(\frac{c}{2}\right) n^2, & \text{if } a = 2, b = -1. \end{cases}$$

The roots of the characteristic equation could involve imaginary numbers. For example, the roots of $x^2 = -1$ are $\pm\sqrt{-1}$. We will not consider examples of this type.

Example 1. Consider the DS $A_{n+2} = -A_{n+1} + 6A_n$. We have $a = -1$, $b = 6$ and $c = 0$, so $a + b \neq 1$. The characteristic equation is $x^2 = -x + 6$. The roots are -3 and 2. Since $r \neq s$, the general solution is

$$A_n = c_1 2^n + c_2(-3)^n. \tag{13}$$

Suppose the IC are $A_0 = 7$, $A_1 = -6$. We want to find c_1 and c_2. Letting $n = 0$ in (13) gives

$$7 = A_0 = c_1 + c_2.$$

Letting $n = 1$ in (13) gives

$$-6 = A_1 = 2c_1 - 3c_2.$$

We now have the following system

$$\begin{aligned} 7 &= A_0 = c_1 + c_2 \\ -6 &= A_1 = 2c_1 - 3c_2. \end{aligned}$$

Solving the system gives $c_1 = 3$, $c_2 = 4$. The particular solution is therefore

$$A_n = 3 \cdot 2^n + 4(-3)^n.$$

Example 2. For

$$A_{n+2} = 6A_{n+1} - 9A_n + 2,$$

we have $a = 6$, $b = -9$, and $c = 2$, so $a + b \neq 1$. The characteristic equation is

$$x^2 = 6x - 9$$

which has roots $r = s = 3$. The general solution is

$$A_n = (c_1 + c_2 n)3^n + \frac{1}{2}.$$

Example 3. For

$$A_{n+2} = 3A_{n+1} - 2A_n + 5,$$

we have $a = 3$, $b = -2$, and $c = 5$, so $a + b = 1$ and the general solution is

$$A_n = c_1 2^n + c_2 - 5n.$$

Example 4. For

$$A_{n+2} = 2A_{n+1} - A_n + 3,$$

$a = 2$, $b = -1$ and $c = 3$, so the general solution is

$$A_n = c_1 n + c_2 + \left(\frac{3}{2}\right) n^2.$$

Problems for Section 10.

1. Find the general solution to

$$A_{n+2} = 5A_{n+1} - 6A_n + 8.$$

Find the particular solution satisfying the IC $A_0 = 1, A_1 = 2$.

2. Find the general solution to

$$A_{n+2} = 2A_{n+1} - A_n + 4.$$

Find the particular solution satisfying $A_0 = 3, A_1 = 6$.

3. Find the general solution to $A_{n+2} = 4A_{n+1} - 4A_n + 4$.

4. Find the general solution to $A_{n+2} = -1A_{n+1} + 2A_n + 3$.

5. Suppose that $a + b \neq 1$ and $r \neq s$, where r, s are the solutions to $x^2 = ax + b$. Show by substitution that the formula

$$A_n = c_1 r^n + c_2 s^n + \frac{c}{1 - a - b}$$

is a solution to the DS $A_{n+2} = aA_{n+1} + bA_n + c$.

6. Suppose $a = 1, b = 2, c = 3, A_0 = 1, A_1 = 2$. Find A_2 and A_{20}, where $A_{n+2} = aA_{n+1} + bA_n + c$.

11 A Model for the National Economy

Let

T = total national income,

C = consumer expenditures,

I = private investment, and

G = government expenditures.

Each of these quantities is measured at discrete times $n = 0, 1, 2 \ldots$. For each n, we assume

$$T_n = C_n + I_n + G_n. \tag{14}$$

Assume that money is spent one time period after it is earned, so that

$$C_{n+1} = mT_n, \qquad m > 0. \tag{15}$$

The proportionality constant m is called the **marginal propensity to consume** (MPC).

Assume that private investment is proportional to the change in consumption, so that

$$I_{n+1} = l(C_{n+1} - C_n), \qquad l > 0. \tag{16}$$

The constant l is called the **accelerator**. As consumption increases, more factories must be built to provide more goods, and investment is needed to build the factories.

Assume that G_n is constant. Think of this as saying that government expenditures are constant in inflation-adjusted dollars. In addition, we will use this constant amount G_n as our unit of money, so that

$$G_n = 1 \tag{17}$$

and all quantities are being measured relative to G_n.

Letting $n \to n + 2$ in (14) and using $G_{n+2} = 1$ gives

$$T_{n+2} = C_{n+2} + I_{n+2} + 1. \tag{18}$$

Equation (16) gives

$$I_{n+2} = l(C_{n+2} - C_{n+1}),$$

so (18) becomes

$$\begin{aligned} T_{n+2} &= C_{n+2} + l(C_{n+2} - C_{n+1}) + 1 \\ &= (1 + l)C_{n+2} - lC_{n+1} + 1. \end{aligned} \tag{19}$$

Equation (15) gives $C_{n+2} = mT_{n+1}$ and $C_{n+1} = mT_n$, so (19) becomes

$$T_{n+2} = m(1 + l)T_{n+1} - mlT_n + 1. \tag{20}$$

The second order linear DS (20) is a model for the national economy.

Example. Suppose $m = \frac{2}{3}$ and $l = \frac{1}{4}$. The model is

$$T_{n+2} = \frac{5}{6}T_{n+1} - \frac{1}{6}T_n + 1.$$

The general solution is

$$T_n = c_1 \left(\frac{1}{2}\right)^n + c_2 \left(\frac{1}{3}\right)^n + 3. \tag{21}$$

Suppose $C_0 = 3$, $I_0 = 2$. We always have $G_0 = 1$, so

$$T_0 = C_0 + I_0 + G_0 = 3 + 2 + 1 = 6.$$

Also, $C_1 = \frac{2}{3}T_0 = 4$ and $I_1 = \frac{1}{4}(C_1 - C_0) = \frac{1}{4}$, so

$$T_1 = C_1 + I_1 + G_1 = 4 + \frac{1}{4} + 1 = \frac{21}{4}.$$

The initial conditions are therefore $T_0 = 6$, $T_1 = \frac{21}{4}$. To find c_1 and c_2, let $n = 0, 1$ in (21) to get

$$\begin{aligned} 6 &= T_0 = c_1 + c_2 + 3 \\ \frac{21}{4} &= T_1 = \frac{1}{2}c_1 + \frac{1}{3}c_2 + 3. \end{aligned}$$

The solution to this linear system is $c_1 = \frac{15}{2}$, $c_2 = -\frac{9}{2}$, so the particular solution that satisfies the initial conditions is

$$T_n = \frac{15}{2}\left(\frac{1}{2}\right)^n - \frac{9}{2}\left(\frac{1}{3}\right)^n + 3,$$

After 10 time units, for example, the national income is

$$T_{10} = \frac{15}{2}\left(\frac{1}{2}\right)^{10} - \frac{9}{2}\left(\frac{1}{3}\right)^{10} + 3 \approx 3.00725.$$

As $n \to \infty$, $T_n \to 3$.

Problems for Section 11.

1. Suppose the MPC is $\frac{1}{2}$ and the accelerator is $\frac{1}{6}$.

a. Write the DS for this national income model.

b. Find the general solution.

c. Say $C_0 = 4, I_0 = 5$. Show that $T_0 = 10$ and $T_1 = \frac{37}{6}$.

d. Find the particular solution that satisfies the IC $T_0 = 10$, $T_1 = \frac{37}{6}$.

e. Find T_8, the national income after 8 time periods.

f. Verify by substitution that your solution in **d.** satisfies the DS.

g. Verify by substitution that your solution in **b.** satisfies the DS.

12 Fixed Points and Stability for Second Order Dynamical Systems

12.1 Fixed Points

Definition. *A* **fixed point** *of a second order DS is a number k so that if $A_n = k$ for $n = 0, 1, 2, \dots$, then A_n is a solution to the DS.*

Consider the second order linear DS

$$A_{n+2} = aA_{n+1} + bA_n + c.$$

Suppose that $a + b \neq 1$. If k is a fixed point, then $A_{n+2} = A_{n+1} = A_n = k$ for all n, so $k = ak + bk + c$ and

$$k = \frac{c}{1 - a - b}.$$

If $a + b = 1$, then the DS is

$$A_{n+2} = aA_{n+1} + (1 - a)A_n + c.$$

If k is a fixed point, then

$$k = ak + (1 - a)k + c = k + c$$

which has a solution if and only if $c = 0$. Therefore, there are no fixed points if $c \neq 0$ and every point is a fixed point if $c = 0$.

12.2 Stability

Definition. *The second order linear DS is* **stable** *if $\lim_{n\to\infty} A_n$ exists for all initial conditions. The DS is* **unstable** *if $\lim_{n\to\infty} |A_n| = \infty$ for some initial conditions.*

There exist DS which are neither stable nor unstable.

12.3 The case $a + b \neq 1$

There is a unique fixed point in this case. If the A_n approach the fixed point for all possible initial conditions, the system is stable and we say that the fixed point is **attracting**. If $|A_n| \to \infty$ for some initial conditions, then the system is unstable and we say that the fixed point is **repelling**. It is also possible that neither one of these two behaviors is present.

Example 1. If the DS is

$$A_{n+2} = \frac{5}{6} A_{n+1} - \frac{1}{6} A_n + 1,$$

then the fixed point is 3 and the general solution is

$$A_n = c_1 \left(\frac{1}{2}\right)^n + c_2 \left(\frac{1}{3}\right)^n + 3.$$

As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$ and $\left(\frac{1}{3}\right)^n \to 0$, so $A_n \to 3$. The fixed point is attracting and the system is stable.

Example 2. If the DS is

$$A_{n+2} = 5A_{n+1} - 6A_n + 2,$$

then the fixed point is 1 and the general solution is

$$A_n = c_1 2^n + c_2 3^n + 1.$$

As $n \to \infty$, $2^n \to \infty$ and $3^n \to \infty$, so $A_n \to \infty$ if c_1 or c_2 is nonzero. The fixed point is repelling and the system is unstable.

Example 3. If the DS is

$$A_{n+2} = \frac{5}{2} A_{n+1} - A_n + 2,$$

then the fixed point is -4 and the general solution is

$$A_n = c_1 \left(\frac{1}{2}\right)^n + c_2 2^n + \frac{4}{5}.$$

As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$ and $2^n \to \infty$, so $A_n \to \infty$ if $c_2 \neq 0$. The fixed point is repelling and the system is unstable.

Example 4. If the DS is

$$A_{n+2} = -\frac{1}{2}A_{n+1} + \frac{1}{2}A_n + 1,$$

then the fixed point is 1 and the general solution is

$$A_n = c_1 \left(\frac{1}{2}\right)^n + c_2 (-1)^n + 1.$$

As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$ and $(-1)^n$ alternates between -1 and 1, so the A_n have no limit as $n \to \infty$ if $c_2 \neq 0$. Also, $|A_n| \not\to \infty$, so the system is neither stable nor unstable.

Example 5. If the DS is

$$A_{n+2} = A_{n+1} - \frac{1}{4}A_n + 1,$$

then the fixed point is 4 and the general solution is

$$A_n = (c_1 + c_2 n)\left(\frac{1}{2}\right)^n + 4.$$

As $n \to \infty$, $A_n \to 0$, so the system is stable.

Given any particular problem, first calculate the fixed point. Then find the general solution and examine its behavior as $n \to \infty$ to determine if the system is stable and the fixed point is attracting (or not). The general case of $a + b \neq 1$ is treated in the appendix to Section 12.

12.4 The case $a + b = 1$

We are not concerned with fixed points being attracting or repelling in this case and examine only the long term behavior of the general solution A_n.

Example 6. If the DS is

$$A_{n+2} = 6A_{n+1} - 5A_n,$$

then the general solution is

$$A_n = c_1 5^n + d.$$

If $c_1 \neq 0$, then $|A_n| \to \infty$, so the system is unstable.

Example 7. If the DS is

$$A_{n+2} = 2A_{n+1} - A_n + 6,$$

then the general solution is

$$A_n = c_1 + c_2 n + 3n^2.$$

The system is unstable since $|A_n| \to \infty$.

Example 8. If the DS is

$$A_{n+2} = A_n,$$

the general solution is

$$A_n = c_1(-1)^n + c_2.$$

and the system is neither stable nor unstable.

Examle 9. If the DS is

$$A_{n+2} = \frac{3}{2}A_{n+1} - \frac{1}{2}A_n,$$

the general solution is

$$A_n = c_1 \left(\frac{1}{2}\right)^n + c_2$$

and the system is stable.

The general case of $a + b = 1$ is treated in the Appendix to Section 12.

Problems for Section 12.

1. For each of the following DS, find the fixed points and discuss the stability of the system.

a. $A_{n+2} = 4A_{n+1} - 4A_n + 4$

b. $A_{n+2} = -\frac{2}{3}A_{n+1} + \frac{1}{3}A_n + 1$

c. $A_{n+2} = 2A_{n+1} - \frac{63}{64}A_n + \frac{1}{64}$

d. $A_{n+2} = -2A_{n+1} - A_n + 2$

e. $A_{n+2} = \frac{1}{2}A_{n+1} - \frac{3}{64}A_n + 35$

f. $A_{n+2} = 7A_{n+1} - 12A_n + 6$

g. $A_{n+2} = \frac{2}{3}A_{n+1} - \frac{1}{9}A_n + 4$

2. This problem concerns the case $a + b = 1$. For each example, determine if there are fixed points and discuss the stability of the system.

a. $A_{n+2} = 3A_{n+1} - 2A_n$

b. $A_{n+2} = A_{n+1} + 2$

c. $A_{n+2} = A_n$

d. $A_{n+2} = \frac{1}{3}A_{n+1} + \frac{2}{3}A_n$

e. $A_{n+2} = 2A_{n+1} - A_n + 2$

f. $A_{n+2} = 2A_{n+1} - A_n$

3. Show that if $a + b \neq 1$ and r is a root of the characteristic equation $x^2 = ax + b$, then $r \neq 1$.

Appendix to Section 12

Now we address the general situation. First, assume that $a + b \neq 1$ and the roots r, s satisfy $r \neq s$. The general solution is

$$A_n = c_1 r^n + c_2 s^n + \frac{c}{1 - a - b}.$$

There are various cases that must be considered.

Case 1. If $|r| < 1$ and $|s| < 1$, then $r^n \to 0$ and $s^n \to 0$ as $n \to \infty$, so

$$A_n \to \frac{c}{1 - a - b}$$

for all initial conditions. The fixed point is attracting and the system is stable.

Case 2. If $|r| > 1$ or $|s| > 1$, then $|A_n| \to \infty$ as $n \to \infty$ for some initial conditions, so the fixed point is repelling and the system is unstable.

Case 3. Suppose $|r| < 1$ and $|s| = 1$, as in Example 4. Since we are assuming that s is a real number, we must have $s = \pm 1$. But $a + b \neq 1$ implies that $s \neq 1$ (see Problem 3), so $s = -1$ and the general solution is

$$A_n = c_1 r^n + c_2 (-1)^n + \frac{c}{1 - a - b}.$$

As $n \to \infty$, $r^n \to 0$ and $(-1)^n = \pm 1$, so the A_n have no limit as $n \to \infty$ if $c_2 \neq 0$. Also, $|A_n| \not\to \infty$, so the system is neither stable nor unstable.

Case 4. Suppose $|r| = |s| = 1$, still with $r \neq s$. From equation (**??**), we have $a + b = r + s - rs$. Since $r \neq s$, $(r, s) = (1, -1)$ or $(r, s) = (-1, 1)$. In each of these cases, we have $a + b = 1$, which is false. Therefore, this case cannot arise.

Now suppose that $a + b \neq 1$ and $r = s$. The general solution is

$$A_n = (c_1 + c_2 n) r^n + \frac{c}{1 - a - b}.$$

Again, various cases can arise.

Case 5. If $|r| < 1$, then $r^n \to 0$ and $nr^n \to 0$ (by L'Hopital's Rule), so

$$A_n \to \frac{c}{1-a-b}$$

for all initial conditions. The fixed point is attracting and the system is stable.

Case 6. If $|r| > 1$, then $|r^n| \to \infty$ and $|nr^n| \to \infty$, so $|A_n| \to \infty$ unless $c_1 = c_2 = 0$. The fixed point is repelling and the system is unstable.

Case 7. Suppose $r = \pm 1$. If $r = s = 1$, then from equation (**??**), $a + b = 1$, which is false. Therefore, $r = s = -1$. The general solution is

$$A_n = (c_1 + c_2 n)(-1)^n + \frac{c}{1-a-b}.$$

If $c_2 \neq 0$, then $|A_n| \to \infty$, so the the fixed point is repelling and the system is unstable.

Now assume $a + b = 1$. There are various cases to consider.

Case 8. Suppose $a \neq 2$ and $c = 0$. The general solution is

$$A_n = c_1(a-1)^n + c_2.$$

If $|a - 1| < 1$, then $A_n \to c_2$ so the system is stable. If $|a - 1| > 1$ and $c_1 \neq 0$, then $|A_n| \to \infty$, so the system is unstable. If $|a - 1| = 1$, then $a = 0$ and $A_n = c_1(-1)^n + c_2$. The system is neither stable nor unstable.

Case 9. Suppose $a \neq 2$ and $c \neq 0$. The general solution is

$$A_n = c_1(a-1)^n + c_2 + \left(\frac{c}{2-a}\right) n.$$

Since $|A_n| \to \infty$, the system is unstable.

Case 10. Suppose $a = 2$, $b = -1$. The general solution is

$$A_n = c_1 + c_2 n + \left(\frac{c}{2}\right) n^2.$$

If $c_2 \neq 0$, then $|An| \to \infty$, so the system is unstable.

13 Stability of Economic Systems

13.1 Discrete Market Models

Recall from Section 9.3 that a discrete market model is given by

$$\begin{aligned} D_t &= S_t \\ D_t &= a - bP_t \\ S_t &= -c + dP_{t-1}, \end{aligned}$$

where a, b, c, d are positive. P_t is the equilibrium price at time $t = 0, 1, 2, \ldots$. The model leads to the first order DS

$$P_{t+1} = \left(-\frac{d}{b}\right) P_t + \frac{a+c}{b},$$

which has the general solution

$$P_t = k \left(-\frac{d}{b}\right)^t + \overline{P},$$

where

$$\overline{P} = \frac{a+c}{b+d}$$

is the fixed point. If $\frac{d}{b} < 1$, then $P_t \to \overline{P}$ as $t \to \infty$ and the system is stable. If $\frac{d}{b} > 1$, then $|P_t| \to \infty$ and the system is unstable. If $\frac{d}{b} = 1$, the system oscillates. These models are easy to understand since there is a simple criteria for convergence in terms of b and d.

Example. Suppose the market is unstable and we would like to convert it to a stable market. The parameters a and c do not affect whether the market is stable or not. To achieve stability, we can either increase b or decrease d (or both) until $\frac{d}{b}$ is less than one. We might do this by convincing consumers to be more sensitive to price, which would increase b, or by offering incentives to decrease supply at a given price level, which would decrease d.

Example. Suppose that we control the market to the extent that we can set the parameters to be whatever we like. Suppose we want the equilibrium prices P_t to converge to 3 as $t \to \infty$. How should we choose a, b, c and d? We must have

$$\overline{P} = \frac{a+c}{b+d} = 3,$$

and to have convergence, we need $\frac{d}{b} < 1$. If we choose $b = 2, d = 1$ to have convergence, then we need $\frac{a+c}{3} = 3$, or $a + c = 9$. We could choose $a = 5, c = 4$, and then we have the system

$$\begin{aligned} D_t &= S_t \\ D_t &= 5 - 2P_t \\ S_t &= -4 + P_{t-1} \end{aligned}$$

that behaves in the desired way.

13.2 National Income Models

Deciding which choice of parameters leads to convergence is more complicated for the second order DS arising from the model for the national economy:

$$T_{n+2} = m(1+l)T_{n+1} - mlT_n + 1,$$

where m is the MPC and l is the accelerator. Another problem is that certain choices of m and l lead to imaginary roots of the characteristic equation. We will not deal with imaginary roots.

Example. Suppose we want the national income to converge to 5 as $n \to \infty$. How should we choose m and l? To have a fixed point of 5, we need

$$\frac{1}{1 - m(1+l) + ml} = 5,$$

which simplifies to $\frac{1}{1-m} = 5$, or $m = \frac{4}{5}$. We will have convergence if the roots of the characteristic equation

$$x^2 - \frac{4}{5}(1+l)x + \frac{4}{5}l = 0$$

satisfy $|r| < 1, |s| < 1$. The roots are given by

$$s = \frac{1}{2}\left(\frac{4}{5}(1+l) - \sqrt{\frac{16}{25}(1+l)^2 - \frac{16}{5}l}\right),$$

$$r = \frac{1}{2}\left(\frac{4}{5}(1+l) + \sqrt{\frac{16}{25}(1+l)^2 - \frac{16}{5}l}\right).$$

If we choose $l = \frac{1}{5}$, for example, then the roots are $s = \frac{6}{25}$ and $r = \frac{18}{25}$. With $m = \frac{4}{5}$ and $l = \frac{1}{5}$, we have $T_n \to 5$ as $n \to \infty$. Now let us find all the choices for l that lead to real roots $r \neq s$ with $|r| < 1$ and $|s| < 1$ when $m = \frac{4}{5}$. The roots are real and distinct when

$$\frac{16}{25}(1+l)^2 - \frac{16}{5}l > 0,$$

which is equivalent to $l^2 - 3l + 1 > 0$. This inequality holds when l is not in the interval

$$\left[\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right] \approx [.381, 2.61].$$

Now we determine which l not in this interval lead to a stable system. Choose r, s so that $|r| > |s|$. It is then sufficient to show that $|r| < 1$. If $l > \frac{3+\sqrt{5}}{2}$, then

$$|r| = \frac{1}{2}\left(\frac{4}{5}(1+l) + \sqrt{\frac{16}{25}(1+l)^2 - \frac{16}{5}l}\right) > \frac{2}{5}(1+l) > 1.$$

Now assume $0 < l < \frac{3-\sqrt{5}}{2}$. Then $|r| < 1$ if and only if

$$\sqrt{\frac{16}{25}(1+l)^2 - \frac{16}{5}l} < 2 - \frac{4}{5}(1+l) = \frac{6}{5} - \frac{4}{5}l,$$

which, after squaring both sides and simplifying, is equivalent to $16 < 36$, which is true. This shows that all l such that $0 < l < \frac{3-\sqrt{5}}{2}$ lead to a stable system when $m = \frac{4}{5}$.

Now we turn to the general case for $m > 0$, $l > 0$. The characteristic equation for the national income model is

$$x^2 - (m + l)x + ml = 0.$$

The roots are

$$x = \frac{1}{2}\left(m(1 + l) \pm \sqrt{m^2(1 + l)^2 - 4ml}\right).$$

The roots are real when

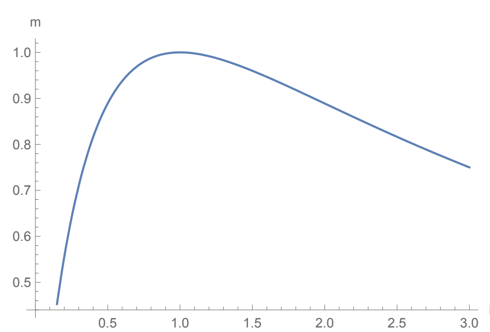
$$m^2(1 + l)^2 - 4ml \geq 0,$$

which is equivalent to

$$m \geq \frac{4l}{(1 + l)^2}.$$

Consider the curve in the $l - m$ plane given by the equation

$$m = \frac{4l}{(1 + l)^2}.$$



A point (l, m) in the first quadrant leads to real roots when (l, m) lies on or above the curve.

Theorem. *If (l, m) lies in the first quadrant and $m \geq \frac{4l}{(1+l)^2}$ then the model is stable if and only if $l < 1$ and $m < 1$.*

Proof. See the Appendix to Section 13. □

In particular, the system is unstable if $m \geq 1$, regardless of the value of l. By equation (15) in Section 11, this occurs if the consumption at any time is greater than or equal to the income at the previous time. If $m < 1$, the system is stable for some $l > 1$, but the only cases for which this occurs have imaginary roots. If $m < 1$, the system is unstable if

$$l \geq \frac{2}{m} - 1 + \frac{2}{m}\sqrt{1 - m}. \tag{22}$$

This shows that if investment is too large a multiple of the change in consumption, then the system is unstable.

Problems for Section 13.

1. Give an example of a discrete market model in which the equilibrium prices converge to 5.

2. We have a national income model with $l = \frac{1}{2}$.

a. Which values of m lead to distinct real roots?

b. Which values of m lead to distinct real roots and a convergent system?

3. We have a national income model with $m = \frac{1}{2}$.

a. Show that we obtain distinct real roots if l is not in the interval $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$.

b. Show that if $0 < l < 3 - 2\sqrt{2}$, then the system is stable.

c. Show that if $l > 3 + 2\sqrt{2}$, then the system is unstable.

4. Suppose that $m = \frac{1}{3}$. Which values of l produce a stable system with real roots?

5. Show that the system is unstable if $m < 1$ and

$$l \geq \frac{2}{m} - 1 + \frac{2}{m}\sqrt{1 - m}.$$

Appendix to Section 13

Using the notation of equation (11) in Section 10, we have $a = m(1+l)$ and $b = -ml$, so $a + b = m$.

Theorem. *If (l, m) lies in the first quadrant and $m = \frac{4l}{(1+l)^2}$ then the model is unstable.*

Proof. If $m \neq 1$, the general solution is

$$T_n = (c_1 + c_2 n)\left(\frac{2l}{1+l}\right)^n + \frac{1}{1-m}.$$

If $c_2 \neq 0$, then $|T_n| \to \infty$, so the model is unstable. If $m = 1$, then $l = 1$, so $a = m(1+l) = 2$ and the general solution is

$$T_n = c_1 + c_2 n + \frac{1}{2}n^2.$$

Since $|T_n| \to \infty$, the model is unstable. □

Assume now that $m > \frac{4l}{(1+l)^2}$. This gives distinct real roots r, s. Assume

$$r = \frac{1}{2}\left(m(1+l) + \sqrt{m^2(1+l)^2 - 4ml}\right), \; s = \frac{1}{2}\left(m(1+l) - \sqrt{m^2(1+l)^2 - 4ml}\right),$$

so that $0 < s < r$.

Lemma 1. *If $m > \frac{4l}{(1+l)^2}$ and $m = 1$, the model is unstable.*

Proof. If $m = 1$, then $a + b = 1$ and $a - 1 = l < 1$ so the general solution is

$$T_n = c_1 l^n + c_2 + \frac{n}{1-l},$$

and the model is unstable since $\frac{n}{1-l} \to \infty$. □

Lemma 2. *If $m > \frac{4l}{(1+l)^2}$ and $m \neq 1$, the model is unstable if $r > 1$.*

Proof. If $m \neq 1$, the general solution is

$$T_n = c_1 r^n + c_2 s^n + \frac{1}{1-m},$$

so if $r > 1$, then $|T_n| \to \infty$ if $c_1 \neq 0$. □

Lemma 3. *If* $m > \frac{4l}{(1+l)^2}$ *and* $l > 1$*, then* $r > 1$*.*

Proof.

$$r = \frac{1}{2}\left(m(1+l) + \sqrt{m^2(1+l)^2 - 4ml}\right) > \frac{m(1+l)}{2} > \frac{4l}{(1+l)^2}\frac{1+l}{2} = \frac{2l}{1+l} > 1$$

since $l > 1$. □

Lemma 4. *Assume* $m > \frac{4l}{(1+l)^2}$*. Then*

(1) if $m(1+l) \geq 2$*, then* $r > 1$*.*

(2) if $m(1+l) < 2$*, then* $m < 1 \iff r < 1$*, and* $m > 1 \iff r > 1$*.*

Proof.

(1) $m(1+l) \geq 2 \implies m(1+l) + \sqrt{m^2(1+l)^2 - 4ml} > 2 \implies r > 1$.

(2) Since $2 - m(1+l) > 0$,

$$\begin{aligned} r < 1 &\iff m(1+l) + \sqrt{m^2(1+l)^2 - 4ml} < 2 \\ &\iff \sqrt{m^2(1+l)^2 - 4ml} < 2 - m(1+l) \\ &\iff m^2(1+l)^2 - 4ml < 4 - 4m(1+l) + m^2(1+l)^2 \iff m < 1. \end{aligned}$$

Similarly, $r > 1 \iff m > 1$. □

Theorem. *If* (l, m) *lies in the first quadrant and* $m > \frac{4l}{(1+l)^2}$ *then the model is stable if and only if* $m < 1$ *and* $l < 1$*.*

Proof. If $m = 1$, the model is unstable by Lemma 1. If $l > 1$, the model is unstable by Lemma 3. Assume now that $m \neq 1$ and $l \leq 1$. If $l = 1$ then $m > 1$, so $m(1+l) > 2$ and the model is unstable by (1) of Lemma 4. If $m < 1$ and $l < 1$, then $m(1+l) < 2$ so by (2) of Lemma 4, the model is stable. Suppose $m > 1$ and $l < 1$. Then if $m(1+l) \geq 2$, (1) of Lemma 4 shows that the model is unstable. If $m(1+l) < 2$, then $m > 1$ implies $r > 1$ by (2) of Lemma 4, so the model is unstable. □

14 Critical Points and Second Derivative Test for Functions of Several Variable

Let $\mathbb{R}^n$ be the set of all points $P = (x_1, \ldots, x_n)$, with each x_i a real number. $\mathbb{R}^n$ is called n-dimensional space. Consider a function of n variables $f(P) = f(x_1, \ldots, x_n)$. For each P, $f(P)$ is a real number. For example,

$$f(x, y) = x^2y - xy^3$$

is a function of two variables, and

$$f(x, y, z, w) = 3xyzw$$

is a function of four variables. We will assume that our functions are nicely behaved and satisfy all necessary assumptions. Let

$$f_i(P) = \frac{\partial f}{\partial x_i}(P), \qquad i = 1, 2, \ldots, n.$$

This is the partial derivative of f with respect to the i-th variable x_i, evaluated at the point P. We will also write $f_{x_i}(P) = f_i(P)$. This derivative is calculated by fixing all the variables other than x_i and differentiating with respect to x_i. For example, if $f(x, y, z) = x^2y + 2xyz$, then

$$\begin{aligned} f_x(x, y, z) &= f_1(x, y, z) = 2xy + 2yz, \\ f_y(x, y, z) &= f_2(x, y, z) = x^2 + 2xz, \\ f_z(x, y, z) &= f_3(x, y, z) = 2xy. \end{aligned}$$

Theorem. *If f has a local maximum or a local minimum at a point P in $\mathbb{R}^n$, then*

$$f_i(P) = 0, \ i = 1, 2, \ldots, n. \tag{23}$$

A point P that satisfies the conditions (23) is called a **critical point** of f. There may exist critical points where f does not have a local extrema. For example, if $f(x) = x^3$, then 0 is a critical point but is not a local extrema for f. To locate the local extrema, we should find the critical points and then check each one using the **second derivative test**.

The second derivatives are defined by

$$f_{ij}(P) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(P), \qquad i, j = 1, 2, \ldots, n.$$

We will also write $f_{x_i x_j}(P) = f_{ij}(P)$. For example if $f(x,y,z) = x^2y + 2xyz$ as above, then

$$\begin{aligned} f_{xx}(x,y,z) &= f_{11}(x,y,z) = 2y, \\ f_{xy}(x,y,z) &= f_{12}(x,y,z) = 2x + 2z, \\ f_{yx}(x,y,z) &= f_{21}(x,y,z) = 2x + 2z, etc. \end{aligned}$$

Note that $f_{12} = f_{21}$. In general, we will have $f_{ij} = f_{ji}$ for our nicely behaved functions.

Recall the second derivative test for functions of two variables $f(x,y)$.

Second Derivative Test (n=2). *Suppose that (a,b) is a critical point of $f(x,y)$. Let*

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2. \tag{24}$$

Then

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b).

2. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local max at (a,b).

3. If $D(a,b) < 0$, then f has neither a local max nor a local min at (a,b). Such a critical point is called a **saddle point***.*

If $D(a.b) = 0$, the test gives no information; there could be a local max, a local min or a saddle point at (a,b). Note that since $f_{xy} = f_{yx}$, we could also write

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)f_{yx}(a,b). \tag{25}$$

For a function of n variables, we have the n^2 second partial derivatives

$$f_{ij}(P), \quad i,j = 1,2,\ldots,n.$$

We form the $n \times n$ **Hessian** matrix of second derivatives

$$H(P) = \begin{bmatrix} f_{11}(P) & \ldots & f_{1n}(P) \\ \vdots & & \vdots \\ f_{n1}(P) & \ldots & f_{nn}(P) \end{bmatrix}.$$

Consider the following square submatrices of $H(P)$:

$$H_1(P) = \begin{bmatrix} f_{11}(P) \end{bmatrix}, \qquad H_2(P) = \begin{bmatrix} f_{11}(P) & f_{12}(P) \\ f_{21}(P) & f_{22}(P) \end{bmatrix},$$

$$H_3(P) = \begin{bmatrix} f_{11}(P) & f_{12}(P) & f_{13}(P) \\ f_{21}(P) & f_{22}(P) & f_{23}(P) \\ f_{31}(P) & f_{32}(P) & f_{33}(P) \end{bmatrix}, \ldots, H_n(P) = H(P).$$

Let d_k be the determinant $|H_k(P)|$. d_k is called the k-th **principal minor** of $H(P)$. Note that $d_n = |H(P)|$.

Second Derivative Test. *Suppose that P is a critical point of $f(x_1, \ldots, x_n)$. Assume first that $d_n \neq 0$. Then*

(1) If $d_k > 0$ for $k = 1, 2, \ldots, n$, then f has a local minimum at P.

(2) If $d_k < 0$ for k odd and $d_k > 0$ for k even, then f has a local maximum at P.

(3) If neither (1) nor (2) holds, then f has a saddle point at P. (A saddle point is a critical point which is neither a local max nor a local min, just as for $n = 2$).

If $d_n = 0$, then the test provides no information.

Note that if we let $n = 2$, the general Second Derivative Test reduces to the special case stated earlier, since $d_1 = f_{xx}(a, b)$ and $d_2 = D(a, b)$.

Example. Let $f(x, y) = x^2 + xy + y^2 + 2x - 2y + 5$. Then

$$f_x(x, y) = 2x + y + 2, \ f_y(x, y) = x + 2y - 2.$$

The only critical point is $(-2, 2)$. The Hessian is

$$H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$H_1(-2, 2) = [2]$ and

$$H_2(-2, 2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

We have $d_1 = 2$ and $d_2 = 3$, so $(-2, 2)$ gives a local minimum.

Example. Let $f(x, y, z) = x^3 + xy^2 + x^2 + y^2 + 3z^2$. Then $f_x = 3x^2 + y^2 + 2x$, $f_y = 2xy + 2y$ and $f_z = 6z$. The critical points are $(0, 0, 0)$ and $(-\frac{2}{3}, 0, 0)$. The Hessian is

$$H(x, y, z) = \begin{bmatrix} 6x + 2 & 2y & 0 \\ 2y & 2x + 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Therefore,

$$H(0, 0, 0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

We have $d_1 = 2$, $d_2 = 4$ and $d_3 = 24$, so $(0, 0, 0)$ gives a local minimum. Also,

$$H\left(-\frac{2}{3}, 0, 0\right) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

We have $d_1 = -2$, $d_2 = -\frac{4}{3}$ and $d_3 = -\frac{24}{3}$, so $(-\frac{2}{3}, 0, 0)$ gives a saddle point.

Problems for Section 14.

1. For each of the following functions, find the critical points and classify each critical point as a local max, local min or saddle point.

a. $f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$

b. $f(x,y) = x^2 + y^3 - 6xy + 3x + 6y$

c. $f(x,y) = e^{-y}(x^2 - y^2)$

d. $f(x,y,z) = x^2 + y^2 + 2z^2 + xz$

e. $f(x,y,z) = xy + xz + 2yz + \frac{1}{x}$

15 Economic Applications

Example 1. Suppose a company sells two products in competitive markets. The company decides how many of each product they will produce, but they have no control over the price For $i = 1, 2$, let

$$\begin{aligned}
Q_i &= \text{the number of units produced of product } i \\
P_i &= \text{the price per unit of product } i \\
R_i &= P_i Q_i = \text{the revenue from product } i \\
R &= R_1 + R_2 = \text{total revenue} \\
C &= \text{cost of production} \\
\pi &= R - C = \text{profit}
\end{aligned}$$

Assume that

$$C = 2Q_1^2 + Q_1 Q_2 + 2Q_2^2.$$

Then

$$\pi = P_1 Q_1 + P_2 Q_2 - 2Q_1^2 - Q_1 Q_2 - 2Q_2^2.$$

Assume the prices are fixed, so that the profit is a function of Q_1 and Q_2, written as $\pi(Q_1, Q_2)$. The company wishes to choose the values of Q_1 and Q_2 which maximize their profit. To maximize the profit function, we must find the critical points and test them to see which provides a maximum. We have

$$\begin{aligned}
\pi_1 = \pi_{Q_1} &= P_1 - 4Q_1 - Q_2, \\
\pi_2 = \pi_{Q_2} &= P_2 - Q_1 - 4Q_2.
\end{aligned}$$

Setting each of the partial derivatives equal to zero gives the system of equations

$$\begin{aligned}
4Q_1 + Q_2 &= P_1 \\
Q_1 + 4Q_2 &= P_2.
\end{aligned}$$

The solution to the system is

$$Q_1 = \frac{4P_1 - P_2}{15}, \quad Q_2 = \frac{4P_2 - P_1}{15}.$$

To test the critical point, we take the second derivatives. We have $\pi_{11} = -4$, $\pi_{12} = -1$ and $\pi_{22} = -4$. The Hessian is

$$H = \begin{bmatrix} -4 & -1 \\ -1 & -4 \end{bmatrix},$$

so $d_1 = -4$ and $d_2 = 15$. The second derivative test shows that the critical point is a local maximum, and it can be shown that it is also an absolute maximum.

Example 2. Suppose we have a company that produces one product. They sell the product in two markets and have a monopoly in both markets. They are therefore free to charge whatever price they wish. As usual, let

$$\begin{aligned} Q_i &= \text{the number of units produced for market } i \\ P_i &= \text{the price per unit in market } i \\ R_i &= P_i Q_i = \text{the revenue from market } i \end{aligned}$$

We assume that $Q_1 = -3P_1 + 30$ and $Q_2 = -2P_2 + 20$ and that the cost of production is $C = 5Q_1 + 6Q_2$. We would like to maximize the profit $\pi = R_1 + R_2 - C$. We have

$$P_1 = 10 - \frac{1}{3}Q_1 \text{ and } P_2 = 10 - \frac{1}{2}Q_2.$$

Therefore the profit is

$$\begin{aligned} \pi &= P_1Q_1 + P_2Q_2 - C \\ &= (10 - \frac{1}{3}Q_1)Q_1 + (10 - \frac{1}{2}Q_2)Q_2 - 5Q_1 - 6Q_2 \\ &= 5Q_1 - \frac{1}{3}Q_1^2 + 4Q_2 - \frac{1}{2}Q_2^2. \end{aligned}$$

To find the critical points of π, we take the partial derivatives and obtain

$$\pi_1 = 5 - \frac{2}{3}Q_1 \text{ and } \pi_2 = 4 - Q_2.$$

Setting these each to zero gives the one critical point $(\frac{15}{2}, 4)$. The Hessian matrix is

$$H = \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -1 \end{bmatrix}.$$

Taking determinants gives $d_1 = -\frac{2}{3}$, $d_2 = \frac{2}{3}$. Since $d_1 < 0$ and $d_2 > 0$, the critical point is a local maximum.

Example 3. Suppose we have a company that produces 2 products which are sold in the same market. The company has a monopoly and may charge whatever price they wish. Let

$$\begin{aligned} Q_i &= \text{the number of units produced of product } i \\ P_i &= \text{the price per unit of product } i \\ R_i &= P_i Q_i = \text{the revenue from product } i \end{aligned}$$

Assume that

$$Q_1 = 40 - 2P_1 + P_2 \text{ and } Q_2 = 15 + P_1 - P_2. \tag{26}$$

Assume the cost of production is $C = Q_1^2 + Q_1 Q_2 + Q_2^2$. Solving the system (26) for P_1 and P_2 gives

$$P_1 = 55 - Q_1 - Q_2 \text{ and } P_2 = 70 - Q_1 - 2Q_2.$$

Therefore, the profit is

$$\begin{aligned} \pi &= P_1 Q_1 + P_2 Q_2 - C \\ &= 55Q_1 + 70Q_2 - 3Q_1 Q_2 - 2Q_1^2 - 3Q_2^2. \end{aligned}$$

There is one critical point $(8, \frac{23}{3})$. The Hessian matrix is

$$H = \begin{bmatrix} -4 & -3 \\ -3 & -6 \end{bmatrix}.$$

The determinants are $d_1 = -4$, $d_2 = 15$. The critical point is a local maximum.

Problems for Section 15.

1. Using the notation of the above discussion, assume that $P_1 = 5$, $P_2 = 10$ and $C = 2Q_1^2 + 3Q_2^2 + 10$. How many units of each product should the company produce so as to maximize their profit?

2. Again using the notation of the above discussion, assume that $P_1 = 3$, $P_2 = 6$ and $C = 2Q_1^2 + Q_1Q_2 + 4Q_2^2 + 20$. How many units of each product should the company produce so as to maximize their profit?

3. Suppose a company sells one product in 2 markets. For $i = 1, 2$, let

$$
\begin{aligned}
Q_i &= \text{the number of units produced for market } i \\
P_i &= \text{the price per unit in market } i \\
R_i &= P_iQ_i = \text{the revenue from market } i \\
C &= \text{cost of production}
\end{aligned}
$$

Assume $Q_1 = -2P_1 + 40$, $Q_2 = -3P_2 + 48$, $C = 10(Q_1 + Q_2)$.

a. Suppose we decide to sell only in market 1. How many units should we produce to maximize our revenue? What is the maximum revenue? How many units should we produce to maximize our profit? What is the maximum profit? (You should use the second derivative test to be sure you are getting a maximum.)

b. Suppose we decide to sell only in market 2. What is the maximum profit?

c. Suppose we decide to sell in both markets at the same time. How many units should we produce for each market to maximize our total profit? What is the maximum profit? (Don't just assume your answer is the sum of the answers from **a** and **b**.)

4. Suppose a company sells one product in 3 markets. For $i = 1, 2, 3$, let

$$
\begin{aligned}
Q_i &= \text{the number of units produced for market } i \\
P_i &= \text{the price per unit in market } i \\
R_i &= P_i Q_i = \text{the revenue from market } i \\
C &= \text{cost of production}
\end{aligned}
$$

Assume $Q_1 = -3P_1 + 90$, $Q_2 = -2P_2 + 60$, $Q_3 = -2P_3 + 90$ and $C = 4Q_1 + 6Q_2 + 9Q_3$. Find and classify the critical points for the profit function $\pi(Q_1, Q_2, Q_3)$.

5. Assume $Q_1 = -3P_1 + 60$, $Q_2 = -2P_2 + 50$, $Q_3 = -2P_3 + 100$ and $C = Q_1Q_2 + 2Q_1Q_3 + 3Q_2Q_3$. Find and classify the critical points for the profit function $\pi(Q_1, Q_2, Q_3)$.

16 Lagrange Multipliers.

Suppose that if we buy x units of product 1 and y units of product 2, then the utility we receive is

$$f(x, y) = xy + 2x + 2y. \tag{27}$$

Each unit of product 1 costs \$1 and each unit of product 2 costs \$3. We have only \$10 to spend, so we must have

$$x + 3y = 10. \tag{28}$$

The goal is to maximize our utility. One method is to write $x = 10 - 3y$ and substitute this in (27) to get

$$f(x, y) = (10 - 3y)y + 2(10 - 3y) + 2y = -3y^2 + 6y + 20.$$

This is a function of only one variable which we denote by $\phi(y)$. We have

$$\phi'(y) = -6y + 6,$$

so the only critical point of $\phi(y)$ is $y = 1$, and ϕ has a maximum there. If $y = 1$, then $x = 7$ and $f(x, y)$ has a maximum value of $f(7, 1) = 23$.

We can also solve our problem using the method of **Lagrange multipliers**. Suppose we have a function $f(x, y)$ which is restricted to a curve $g(x, y) = c$. The curve $g(x, y) = c$ is called a **constraint**. A point (x, y) which is a local maximum or a local minimum for $f(x, y)$ restricted to $g(x, y) = c$ is called a **constrained extremum**.

Theorem. *Suppose that $f(x, y)$ restricted to $g(x, y) = c$ has an extremum at a point (x, y) where $\nabla g(x, y) \neq 0$. Then*

$$\nabla f(x, y) = \lambda \nabla g(x, y) \tag{29}$$

for some λ.

The points (λ, x, y) which satisfy (29) are called **constrained critical points**.

Applying this method to the example above, we see that

$$\nabla f(x, y) = (y + 2, x + 2), \qquad \nabla g(x, y) = (1, 3),$$

so (29) becomes

$$(y+2, x+2) = \lambda(1,3).$$

x and y must also satisfy $x + 3y = 10$, so we obtain the system of equations

$$\begin{aligned} y + 2 &= \lambda \\ x + 2 &= 3\lambda \\ x + 3y &= 10. \end{aligned}$$

This system has the solution

$$(\lambda, x, y) = (3, 7, 1). \tag{30}$$

We now know that if there is a constrained extremum for $f(x, y)$, then it must occur at $(x, y) = (7, 1)$. It still must be verified that this point leads to a maximum.

We can generalize this approach to the case where $f : \mathbb{R}^n \to \mathbb{R}$ is a function of $\mathbf{x} = (x_1, \ldots, x_n)$. We subject f to k constraints

$$\begin{aligned} g_1(\mathbf{x}) &= c_1 \\ g_2(\mathbf{x}) &= c_2 \\ &\vdots \\ g_k(\mathbf{x}) &= c_k. \end{aligned}$$

Theorem. *Suppose $k < n$. Let S be the set of all points $\mathbf{x}$ in $\mathbb{R}^n$ where*

$$g_1(\mathbf{x}) = c_1, \ldots, g_k(\mathbf{x}) = c_k.$$

If f has an constrained extremum at a point $\mathbf{x}_0$ where $\nabla g_1(\mathbf{x}_0), \ldots, \nabla g_k(\mathbf{x}_0)$ are linearly independent vectors, then there are scalars $\lambda_1, \ldots, \lambda_k$ such that

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \cdots + \lambda_k \nabla g_k(\mathbf{x}_0).$$

Example 1. Let C consist of all the points in $\mathbb{R}^3$ which lie on both the cone $z^2 = x^2 + y^2$ and the plane $z = x + y + 2$. A picture indicates that C consists of two separate pieces. One piece lies on the part of the cone which is above the $x - y$ plane; the other piece lies below the $x - y$ plane. We want to find the points in C which are closest to the origin and the points in C which are farthest from the origin. We therefore want to find the extreme values of the function $f(\mathbf{x}) = \text{dist}(\mathbf{x}, \mathbf{0})$, with $\mathbf{x} = (x, y, z)$. It is easier to find the extreme values of

$$\phi(\mathbf{x}) = f(\mathbf{x})^2 = x^2 + y^2 + z^2.$$

We also have the two constraints

$$g_1(x, y, z) = x^2 + y^2 - z^2 = 0, \qquad g_2(x, y, z) = x + y - z = -2.$$

We have

$$\nabla g_1(x, y, z) = (2x, 2y, -2z), \qquad \nabla g_2(x, y, z) = (1, 1, -1). \tag{31}$$

The equation $\nabla \phi = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ is

$$(2x, 2y, 2z) = \lambda_1(2x, 2y, -2z) + \lambda_2(1, 1, -1). \tag{32}$$

Equation (32) and the two constraint equations (31) produce the system

$$\begin{aligned} 2x &= 2\lambda_1 x + \lambda_2 \\ 2y &= 2\lambda_1 y + \lambda_2 \\ 2z &= -2\lambda_1 z - \lambda_2 \\ x^2 + y^2 - z^2 &= 0 \\ x + y - z &= -2. \end{aligned} \tag{33}$$

We get two possible extrema from the solutions. They are

$$\mathbf{x}_1 = (-2+\sqrt{2}, -2+\sqrt{2}, -2+2\sqrt{2}), \qquad \mathbf{x}_2 = (-2-\sqrt{2}, -2-\sqrt{2}, -2-2\sqrt{2}).$$

We will show below that these are both constrained minima.

Second Derivative Test for Constrained Extrema.

$f : \mathbb{R}^n \to \mathbb{R}$ is a function of $\mathbf{x} = (x_1, \ldots, x_n)$. Supppose that $k < n$ and we have k constraints

$$\begin{aligned} g_1(\mathbf{x}) &= c_1 \\ g_2(\mathbf{x}) &= c_2 \\ &\vdots \\ g_k(\mathbf{x}) &= c_k. \end{aligned}$$

Assume that $\mathbf{x}_0$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ satisfy

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \cdots + \lambda_k \nabla g_k(\mathbf{x}_0).$$

The pair of vectors $(\boldsymbol{\lambda}, \mathbf{x}_0)$ is called a **constrained critical point**. We can formulate a second derivative test for constrained critical points by considering the **Lagrangian**. This is a function of $n + k$ variables defined by

$$L(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \sum_{i=1}^{k} \lambda_i (g_i(x_1, \ldots, x_n) - c_i).$$

We construct the Hessian matrix of this function and evaluate it at the constrained critical point $(\boldsymbol{\lambda}, \mathbf{x}_0)$. We obtain the $(n + k) \times (n + k)$ matrix

$$H_L(\boldsymbol{\lambda}, \mathbf{x}_0) = \begin{bmatrix} 0 & \cdots & 0 & -\frac{\partial g_1}{\partial x_1}(\mathbf{x}_0) & \cdots & -\frac{\partial g_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{\partial g_k}{\partial x_1}(\mathbf{x}_0) & \cdots & -\frac{\partial g_k}{\partial x_n}(\mathbf{x}_0) \\ -\frac{\partial g_1}{\partial x_1}(\mathbf{x}_0) & \cdots & -\frac{\partial g_k}{\partial x_1}(\mathbf{x}_0) & h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_1}{\partial x_n}(\mathbf{x}_0) & \cdots & -\frac{\partial g_k}{\partial x_n}(\mathbf{x}_0) & h_{n1} & \cdots & h_{nn} \end{bmatrix} \tag{34}$$

where

$$h_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0) - \lambda_1 \frac{\partial^2 g_1}{\partial x_j \partial x_i}(\mathbf{x}_0) - \lambda_2 \frac{\partial^2 g_2}{\partial x_j \partial x_i}(\mathbf{x}_0) - \cdots - \lambda_k \frac{\partial^2 g_k}{\partial x_j \partial x_i}(\mathbf{x}_0).$$

This matrix is sometimes called the **bordered Hessian**.

Note that the constrained critical points $(\boldsymbol{\lambda}, \mathbf{x}_0)$ which satisfy

$$\nabla f(\mathbf{x}) = \lambda_1 \nabla g_1(\mathbf{x}) + \cdots + \lambda_k \nabla g_k(\mathbf{x}). \tag{35}$$

are the same as the critical points of the Lagrangian $L(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n)$. If you are planning to test the constrained critical points with the second derivative test, then you should find the constrained critical points by solving $\nabla L = \mathbf{0}$ rather than solving (35). If you solve (35), then you still need to calculate the partial derivatives of L before constructing the Hessian of L, so you end up calculating the partials of f and the g_i twice.

For $j = 1, \ldots, n+k$, let H_j be the $j \times j$ submatrix in the upper left corner of $H_L(\boldsymbol{\lambda}, \mathbf{x}_0)$. Let $d_j = |H_j|$. Calculate the following sequence of numbers:

$$\{(-1)^k d_{2k+1}, (-1)^k d_{2k+2}, \ldots, (-1)^k d_{k+n}\}. \tag{36}$$

Assume $d_{k+n} \neq 0$. Then

(i) If the sequence in (36) consists entirely of positive numbers, then f has a constrained local minimum at $\mathbf{x}_0$.

(ii) If the sequence in (36) alternates as negative, positive, negative, ..., then f has a constrained local maximum at $\mathbf{x}_0$.

(iii) If neither case (i) nor case (ii) holds, then f has a constrained saddle point at $\mathbf{x}_0$.

If $d_{k+n} = 0$, then the test fails.

The sequence in (36) consists of the first term $(-1)^k d_{2k+1}$, the last term $(-1)^k d_{k+n}$ and all the terms in between the two. The first term may equal the last term, in which case the sequence consists of only one term.

Example 2.

a. If $n = 2$ and $k = 1$, then $2k + 1 = 3$ and $n + k = 3$, so the sequence consists only of the term $\{-d_3\}$.

b. If $n = 3$ and $k = 1$, then $2k + 1 = 3$ and $n + k = 4$, so the sequence consists of $\{-d_3, -d_4\}$.

c. If $n = 3$ and $k = 2$, then $2k+1 = 5$ and $n+k = 5$, so the sequence consists only of the term $\{d_5\}$.

Example 3. Consider again $f(x, y) = xy + 2x + 2y$ with the constraint $x + 3y = 10$. The Lagrangian is

$$L(l, x, y) = f(x, y) - l(g(x, y) - c) = xy + 2x + 2y - l(x + 3y - 10). \quad (37)$$

To construct the matrix in (34), we take the second partial derivatives of (37) to get

$$H_L(l, x, y) = \begin{bmatrix} 0 & -1 & -3 \\ -1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}. \quad (38)$$

Recall (30) that the only constrained critical point is $(\lambda, x, y) = (3, 7, 1)$. Substituting this in (38) gives

$$H_L(3, 7, 1) == \begin{bmatrix} 0 & -1 & -3 \\ -1 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}.$$

The sequence (36) is $-d_3 = -6$, showing that $(x, y) = (7, 1)$ gives a constrained local maximum of f.

Example 4. Consider again Example 1. From (33), we get the two constrained critical points

$$(\boldsymbol{\lambda_1}, \mathbf{x}_1) = (2\sqrt{2} - 3, 16\sqrt{2} - 24, \sqrt{2} - 2, \sqrt{2} - 2, 2\sqrt{2} - 2)$$

and

$$(\boldsymbol{\lambda_2}, \mathbf{x}_2) = (-2\sqrt{2} - 3, -16\sqrt{2} - 24, -\sqrt{2} - 2, -\sqrt{2} - 2, -2\sqrt{2} - 2).$$

The Lagrangian is

$$L(l,m,x,y,z) = x^2+y^2+z^2 - l(x^2+y^2-z^2) - m(x+y-z+2)$$

and the Hessian is

$$H_L(l,m,x,y,z) = \begin{bmatrix} 0 & 0 & -2x & -2y & 2z \\ 0 & 0 & -1 & -1 & 1 \\ -2x & -1 & 2-2l & 0 & 0 \\ -2y & -1 & 0 & 2-2l & 0 \\ 2z & 1 & 0 & 0 & 2+2l \end{bmatrix}.$$

The sequence (36) consists of $|H_5|$. A calculation shows that

$$|H_5(\boldsymbol{\lambda_1}, \mathbf{x}_1)| = 128 - 64\sqrt{2} \approx 37.49$$

and

$$|H_5(\boldsymbol{\lambda_2}, \mathbf{x}_2)| = 128 + 64\sqrt{2} \approx 218.51.$$

Since both numbers are positive, $\mathbf{x}_1$ and $\mathbf{x}_2$ are both local constrained minima. But

$$f(\mathbf{x}_1) = 24 - 16\sqrt{2}$$

and

$$f(\mathbf{x}_2) = 24 + 16\sqrt{2},$$

showing that $\mathbf{x}_1$ gives the global minimum. $\mathbf{x}_1$ is the point on the upper part of C which is closest to the origin; $\mathbf{x}_2$ is the point on the lower part which is closest. Since $\mathbf{x}_1$ is closer to the origin than $\mathbf{x}_2$, $\mathbf{x}_1$ must be the global minimum. To see that$f$ does not have a maximum value on C, let x be any number, let

$$y = \frac{-2(x+1)}{(x+2)}$$

and let

$$z = x+y+2.$$

Then (x,y,z) is on the plane $z = x+y+2$. Also, $z^2 = x^2+y^2$, so (x,y,z) is also on the cone. Therefore, (x,y,z) is on C for every x. In addition, $\text{dist}((x,y,z),\mathbf{0})^2$ equals

$$x^2+y^2+z^2 = 2(x^2+y^2) = 2x^2 + 8\left(\frac{x+1}{x+2}\right)^2,$$

which gets arbitrarily large as $x \to \infty$, showing that f has no constrained global maximum.

Problems for Section 16.

1. Find the constrained critical points for each of the following functions with the given constraints.

a. $f(x, y) = 5x + 2y$, $5x^2 + 2y^2 = 14$

b. $f(x, y, z) = xyz$, $2x + 3y + z = 6$

c. $f(x, y, z) = 2x + y^2 - z^2$, $x - 2y = 0$, $x + z = 0$

2. For each function in **1**, classify the constrained critical points using the bordered Hessian and the second derivative test.

17 Optimization with Inequality Constraints

Lagrange multipliers showed how to maximize or minimize a function subject to some constraints, each of which was an equality. We will now allow the constraints to be inequalities.

Example 1. Suppose we have \$5 to spend on apples and bananas. If we spend x on apples and y on bananas, our utility is

$$U(x, y) = xy(9 - x - y).$$

The constraints are $x \geq 0$, $y \geq 0$ and $x + y \leq 5$. The set of possible (x, y) is a triangular region whose boundary has vertices $(0, 0)$, $(5, 0)$ and $(0, 5)$. We want to maximize our utility for (x, y) in this region. If the maximum occurs in the interior of the region, it must occur at a critical point of $U(x, y)$. The equations $U_x = U_y = 0$ for a critical point are

$$\begin{aligned} U_x(x, y) &= y(9 - x - y) - xy = 0, \\ U_y(x, y) &= x(9 - x - y) - xy = 0. \end{aligned}$$

The solutions to this system are $(0, 0)$, $(0, 9)$, $(3, 3)$ and $(9, 0)$. None of these lies in the interior of the region, so we must check the triangular boundary. Consider first the vertical line segment from $(0, 0)$ to $(0, 5)$. For $(0, y)$ on this line, $U(0, y) = 0$. Similarly, on the horizontal line segment from $(0, 0)$ to $(5, 0)$, we have $U(x, 0) = 0$. On the line segment joining $(5, 0)$ and $(0, 5)$, we have

$$U(x, y) = U(x, 5 - x) = x(5 - x)(9 - x - (5 - x)) = 20x - 4x^2.$$

The quantity $20x - 4x^2$, where $0 \leq x \leq 5$, has a maximum value at $x = 2.5$. The maximum value of $U(x, y)$ on this side of the triangle therefore occurs at $(2.5, 2.5)$, and $U(2.5, 2.5) = 25$. The maximum utility subject to the constraints is obtained when we spend \$2.5 on apples and \$2.5 on bananas. We could also have found the maximum of $U(x, y)$ on each of the three line segments by using Lagrange multipliers, since each of the line segments is an equality constraint.

Example 2. The utility function is again $U(x, y) = xy(9 - x - y)$. The constraints are $x \geq 0$, $y \geq 0$ and $x + y \leq 8$. The set of possible (x, y) is a triangular region whose boundary has vertices $(0, 0)$, $(8, 0)$ and $(0, 8)$. The critical point $(3, 3)$ for $U(x, y)$ now lies within the triangular region and $U(3, 3) = 27$. On the line segment joining $(8, 0)$ and $(0, 8)$, the maximum value of $U(x, y)$ occurs at $(4, 4)$, and $U(4, 4) = 16$. The maximum utility therefore occurs at $(x, y) = (3, 3)$.

Example 3. Let $U(x, y) = 10 - x^2 - y^2 + 2y - 2x$. The constraints are $x \geq 0$, $y \geq 0$ and $x + y \leq 5$, just as in Example 1. $U(x, y)$ has the single critical point $(-1, 1)$, which does not lie within the triangular region. The maximum utility therefore occurs on the boundary. On the vertical line segment,

$$U(x, y) = U(0, y) = 10 - y^2 + 2y, 0 \leq y \leq 5.$$

This has a maximum at $y = 1$, and $U(0, 1) = 11$. On the horizontal line segment,

$$U(x, y) = U(x, 0) = 10 - x^2 - 2x, 0 \leq x \leq 5.$$

This has a maximum at $x = 0$, and $U(0, 0) = 10$. On the line segment joining $(5, 0)$ and $(0, 5)$, we have

$$U(x, y) = U(x, 5 - x) = -2x^2 + 6x - 5, 0 \leq x \leq 5.$$

This has a maximum at $x = 1.5$, and $U(1.5, 3.5) = -.5$. The maximum utility on the triangular region is therefore $U(0, 1) = 11$.

Example 4. In Example 1 of Section 15, suppose that $P_1 = 2$ and $P_2 = 3$. The profit function is then

$$\pi(Q_1, Q_2) = 2Q_1 + 3Q_2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2.$$

We want to find the maximum value of $\pi(Q_1, Q_2)$ on the region $Q_1 \geq 0$, $Q_2 \geq 0$. If the the maximum occurs in the interior of the region, it must occur at a critical point. The only critical point is $(\frac{1}{3}, \frac{2}{3})$, and $\pi(\frac{1}{3}, \frac{2}{3}) = \frac{4}{3}$. Now we check the boundary. On the edge $Q_2 = 0$, $Q_1 \geq 0$, we have $\pi(Q_1, Q_2) = 2Q_1 - 2Q_1^2$. The maximum value of this function is $\pi(\frac{1}{2}, 0) = \frac{1}{2}$. On the edge $Q_1 = 0$, $Q_2 \geq 0$, we have $\pi(Q_1, Q_2) = 3Q_2 - 2Q_2^2$. The maximum value of this function is $\pi(0, \frac{3}{4}) = \frac{9}{8}$. Comparing the maximum value in the interior with the maximum values on the edges, we see that the maximum value of the profit is $\frac{4}{3}$ at the critical point.

Problems for Section 17.

1. Maximize $f(x, y) = 2x - x^2 - 4y - y^2$, subject to $x \geq 0$, $y \geq 0$.

2. Maximize $f(x, y) = 2x - x^2 - 4y - y^2$, subject to $x \geq 0$, $y \geq 0$, $3x + y \leq 2$.

3. Maximize $f(x, y) = 2x - x^2 + 5y - y^2 - (x + y)^2$, subject to $x \geq 0$, $y \geq 0$.

4. In Example 2 of Section 15, the profit function is

$$\pi(Q_1, Q_2) = 5Q_1 - \frac{1}{3}Q_1^2 + 4Q_2 - \frac{1}{2}Q_2^2.$$

Maximize the profit, subject to $Q_1 \geq 0$, $Q_2 \geq 0$, $Q_1 + 2Q_2 \leq 16$.

18 Linear Programming

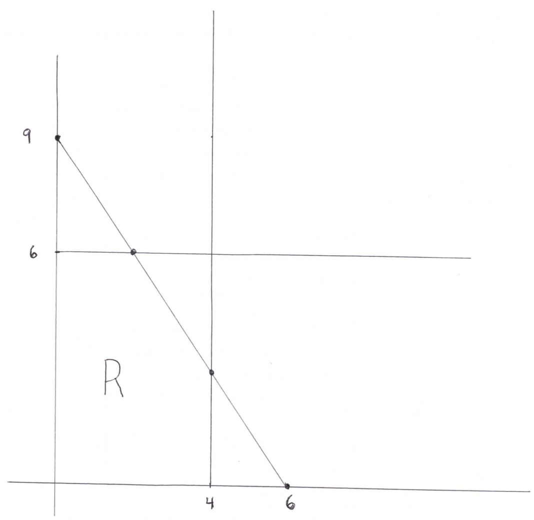
Linear programming is a technique for finding the maximum or minimum of a linear function, called the **objective function**, which is subject to various constraints, each of which is a linear inequality.

Example 1. Suppose that we want to maximize $z = 3x + 5y$, subject to the following constraints:

$$\begin{aligned} &x \leq 4, \\ &2y \leq 12, \\ &3x + 2y \leq 18, \\ &x \geq 0,\ y \geq 0. \end{aligned}$$

This is an example of a linear programming problem. We first graph all the inequalities so as to obtain a clear picture of the values (x, y) which are permitted. The set of these permissible values is called the **feasible region** and is denoted by R. We must locate the point (or points) (x, y) in R which gives the largest value of $z = 3x + 5y$. Suppose we consider the line $10 = 3x + 5y$. This line intersects R and any point on the line produces a z-value of 10. We may slide the line up to the parallel line $20 = 3x + 5y$, which also intersects R, providing points with a z-value of 20. We can continue to slide this line upward, producing parallel lines which intersect R and provide even larger z-values. The largest z-value we obtain is on the line $36 = 3x + 5y$, which intersects R in the single point $(2, 6)$ and produces a z-value of 36. Note that this maximum occurs at what we could call a "corner point" of R.

The **simplex method** is a general procedure for solving linear programming problems. In Example 1, note that the region R is bounded by several lines. These lines are called the **constraint borders**. The points where these lines intersect are called the **corner points**. In the example, the corner points are $(0, 0)$, $(4, 0)$, $(4, 3)$, $(2, 6)$ and $(0, 6)$. The simplex method tells us that the solution to the linear programming problem will occur at a corner point. In some problems there will also be solutions that are not at a corner point. To solve the linear programming problem, simply check each of the corner points to see which one produces the largest (or smallest) value of the objective function. In Example 1, the



corner points produce the following z-values:

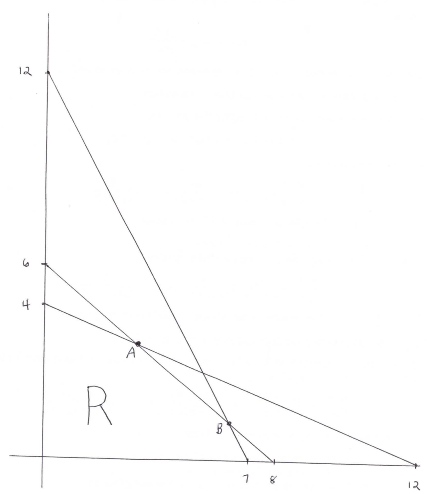
$$(0,0) \to 0,\ (4,0) \to 12,\ (4,3) \to 27,\ (2,6) \to 36,\ (0,6) \to 30.$$

We conclude that the maximum value of the objective function is 36.

Example 2. A company produces 2 products, denoted x and y. The company charges 20 per unit for x and 15 per unit for y. The market is large enough so that the company can sell all that it produces at these prices. The production of each product requires 3 inputs, denoted a, b and c. The company can buy at most 60 units of a, 24 of b and 84 of c. To produce a single unit of x, the company needs 5 units of a, 3 of b and 12 of c. To produce a single unit of y, the company needs 15 units of a, 4 of b and 7 of c. The company wants to maximize the revenue it receives from selling the two products. The linear programming problem can be stated as follows:

$$\begin{aligned} \text{Maximize: } & z = 20x + 15y \\ \text{Subject to: } & 5x + 15y \leq 60, \\ & 3x + 4y \leq 24, \\ & 12x + 7y \leq 84, \\ & x \geq 0,\ y \geq 0. \end{aligned}$$

The five corner points of the feasible region produce the following z-values:



$(0,0) \rightarrow 0,\ (7,0) \rightarrow 140,\ B = (6.22, 1.33) \rightarrow 144.35,\ A = (4.8, 2.4) \rightarrow 132,\ (0,4) \rightarrow 60.$

The maximum possible revenue is therefore 144.35.

Problems for Section 18.

1. Consider the following linear programming problem:

$$\begin{aligned} \text{Maximize: } & z = 3x + 5y \\ \text{subject to: } & x + 3y \leq 15, \\ & 4x + 3y \leq 24, \\ & x \geq 0,\ y \geq 0. \end{aligned}$$

a. Graph the feasible region.

b. Solve the problem.

2. A company produces 2 products, denoted P and Q. The company needs 4 inputs, denoted g, l, s and w to produce the products. The prices of the inputs are: s is .15 per unit, l is .05 per unit, g is .50 per unit and w is free. To produce one unit of P, the company needs 1 units of s, 2 of l, and 1 of w. To produce one unit of Q, the company needs $\frac{2}{3}$ unit of s, 3 of l and 1 of g. The amounts available for the company to use are 10 units of s, 30 of l and 20 of g. One unit of P sells for .4 dollars, and one unit of Q sells for 1 dollar. Use the simplex method to answer the following questions. How many units of each product should the company make in order to maximize their profit? What is the maximum profit?