Problem 1. Solve the differential equation $\frac{dy}{dt} - 2ty = t$ with the initial condition $y(0) = 1$. Verify your solution.

$u = -2t \qquad w = t$

$\Rightarrow \int u \, dt = \int -2t \, dt = -t^2$

$\int w e^{\int u \, dt} dt = \int t e^{-t^2} dt = -\frac{1}{2} e^{-t^2}$

$\Rightarrow y(t) = e^{t^2}\left(A - \frac{1}{2}e^{-t^2}\right) = Ae^{t^2} - \frac{1}{2}$

Initial condition: $1 = y(0) = A - \frac{1}{2} \Rightarrow A = \frac{3}{2}$

$\Rightarrow \boxed{y(t) = \frac{3}{2}e^{t^2} - \frac{1}{2}}$

Verify: $\frac{dy}{dt} - 2ty = 3te^{t^2} - 2t\left(\frac{3}{2}e^{t^2} - \frac{1}{2}\right)$

$= \cancel{3te^{t^2}} - \cancel{3te^{t^2}} + t$

$= t$ ✓

Problem 2. Is the differential equation $(1+t^2)\,dy + 2ty\,dt = 0$ is exact? Why? Solve this differential equation.

$$M = 1+t^2 \qquad N = 2ty$$

$$\frac{\partial M}{\partial t} = 2t \qquad \frac{\partial N}{\partial y} = 2t \quad \Rightarrow \quad \frac{\partial M}{\partial t} = \frac{\partial N}{\partial y} \quad \Rightarrow$$ This differential equation is exact.

① $F(y,t) = \int (1+t^2)\,dy + \varphi(t)$

$\qquad = y + t^2 y + \varphi(t)$

② $\dfrac{\partial F}{\partial t} = 2ty + \dfrac{d\varphi}{dt}$

Since $N = \dfrac{\partial F}{\partial t} \Rightarrow 2ty + \dfrac{d\varphi}{dt} = 2ty \Rightarrow \dfrac{d\varphi}{dt} = 0$

③ $\varphi(t) = \int 0\,dt = k_1$

④ $F(y,t) = y + t^2 y + k_1$

constants

$\Rightarrow$ Solution: $F(y,t) = k_2 \iff y + t^2 y = c$

$\iff \boxed{y(t) = \dfrac{c}{1+t^2}}$

Problem 3. Is the differential equation $\frac{1}{y}\,dy + \frac{2t}{1+t^2}\,dt = 0$ separable? Why? Solve this differential equation.

Let $M = \frac{1}{y}$ and $N = \frac{2t}{1+t^2}$.

This differential equation has separable variables since M is a function of y alone and N is a function of t alone.

To solve: $$\frac{1}{y}\,dy = \frac{-2t}{1+t^2}\,dt$$

$$\Leftrightarrow \int \frac{1}{y}\,dy = \int \frac{-2t}{1+t^2}\,dt$$

$$\Leftrightarrow \ln|y| + k_1 = -\ln|1+t^2| + k_2$$

$$\Leftrightarrow \ln|y| = \ln\left|\frac{1}{1+t^2}\right| + k_3$$

$$\Leftrightarrow |y| = e^{k_3}\left|\frac{1}{1+t^2}\right|$$

$$\Leftrightarrow \boxed{y(t) = \frac{c}{1+t^2}}$$

Problem 4. Solve the differential equation $\dfrac{dy}{dt} = \dfrac{-2ty}{1+t^2}$.

We can rewrite this equation as:

$$(1+t^2)\,dy = -2ty\,dt$$

$$\Longleftrightarrow \quad (1+t^2)\,dy + 2ty\,dt = 0$$

This is the differential equation from Problem 2!

Problem 5. Solve the differential equation $\frac{dy}{dt} - \frac{3}{t}y = t^4 y^{1/3}$.

This is a Bernoulli equation with $R = -\frac{3}{t}$, $T = t^4$, $m = \frac{1}{3}$.

Let $z = y^{2/3}$. $\Rightarrow$ $\frac{dz}{dt} + \frac{2}{3}\left(-\frac{3}{t}\right)z = \frac{2}{3}t^4$

$\Leftrightarrow$ $\frac{dz}{dt} - \frac{2}{t}z = \frac{2}{3}t^4$

Here, $u = -\frac{2}{t}$, $w = \frac{2}{3}t^4$

$\Rightarrow \int u \, dt = \int -\frac{2}{t}\, dt = -2\ln|t|$

$$\int w e^{\int u\,dt}\, dt = \int \frac{2}{3}t^4 e^{\ln|t|^{-2}}\, dt$$

$$= \int \frac{2}{3}t^4 t^{-2}\, dt = \int \frac{2}{3}t^2\, dt$$

$$= \frac{2}{9}t^3$$

$\Rightarrow z(t) = e^{2\ln|t|}\left(A + \frac{2}{9}t^3\right)$

$$= t^2\left(A + \frac{2}{9}t^3\right)$$

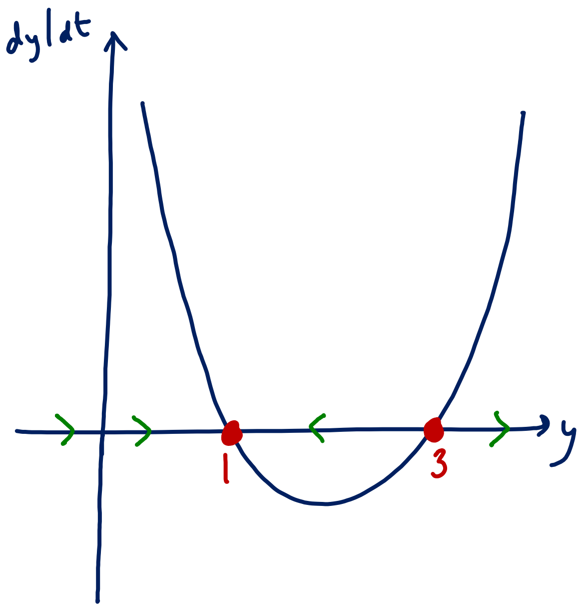
$$= At^2 + \frac{2}{9}t^5$$

$$\Rightarrow \boxed{y(t) = z(t)^{3/2} = \left(At^2 + \frac{2}{9}t^5\right)^{3/2}}$$

Problem 6. Plot the phase line for the differential equation $\frac{dy}{dt} = y^2 - 4y + 3$. What are the equilibrium points? Are they dynamically stable or unstable?

Note that $y^2 - 4y + 3 = (y-3)(y-1)$.

So, the phase line looks like:



There are 2 equilibrium points:

$y = 1$ which is dynamically stable

$y = 3$ which is dynamically <u>un</u>stable.

Problem 7. Solve the difference equation $y_{t+1} - \frac{1}{2}y_t = 3$ with the initial condition $y_0 = 5$. Verify your solution. Is y_t oscillating? Is y_t divergent?

Here, $a = -\frac{1}{2}$, $c = 3$ $\Rightarrow$ $\frac{c}{1+a} = \frac{3}{1-\frac{1}{2}} = 6$

$$\Rightarrow y_t = (5-6)\left(\frac{1}{2}\right)^t + 6$$

$$\Rightarrow \boxed{y_t = -\left(\frac{1}{2}\right)^t + 6}$$

y_t is nonoscillatory, since $b = -a = \frac{1}{2} > 0$.

y_t is convergent, since $|b| < 1$.

Problem 8. Consider the following model of a market with a single product in continuous time. The model variables are

P = unit price $\qquad Q_d$ = quantity demanded $\qquad Q_s$ = quantity supplied

The model equations are:

$$\frac{dP}{dt} = \frac{1}{2}(Q_d - Q_s)$$
$$Q_d = 5 - 2P + \frac{dP}{dt}$$
$$Q_s = -1 + 3P$$

a. Combine these equations to find a single equivalent differential equation.
b. Solve this differential equation.
c. What can you say about the price of the product in the long run?

a. $\frac{dP}{dt} = \frac{1}{2}\left(5 - 2P + \frac{dP}{dt}\right) - \frac{1}{2}(-1 + 3P)$

$\Leftrightarrow \frac{dP}{dt} = \frac{5}{2} - P + \frac{1}{2}\frac{dP}{dt} + \frac{1}{2} - \frac{3}{2}P$

$\Leftrightarrow \frac{1}{2}\frac{dP}{dt} + \frac{5}{2}P = 3$

$\Leftrightarrow \frac{dP}{dt} + 5P = 6$

b. $u = 5, \ w = 6 \Rightarrow \int u \, dt = 5t$

$\int w e^{\int u\,dt} dt = \int 6e^{5t} dt = \frac{6}{5}e^{5t}$

$\Rightarrow P(t) = e^{-5t}\left(A + \frac{6}{5}e^{5t}\right) = Ae^{-5t} + \frac{6}{5}$

c. As $t \to \infty$, $P(t) \to \frac{6}{5}$. If $A > 0$, $P(t)$ decreases towards $\frac{6}{5}$; if $A < 0$, $P(t)$ increases towards $\frac{6}{5}$.

Problem 9. Consider the following version of the Solow growth model. The model variable is

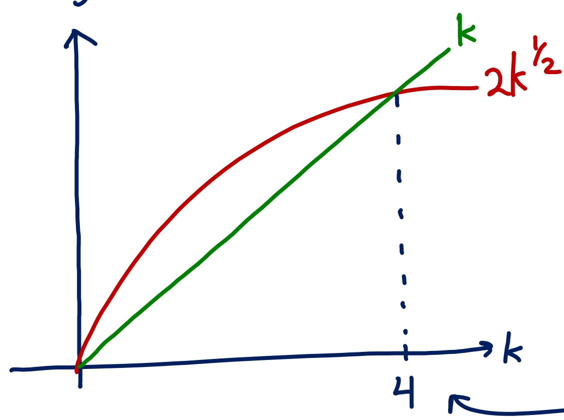
$$k = \text{capital-to-labor ratio}$$

The model equation is

$$\dot{k} = 2k^{1/2} - k \qquad (\text{assuming } k > 0)$$

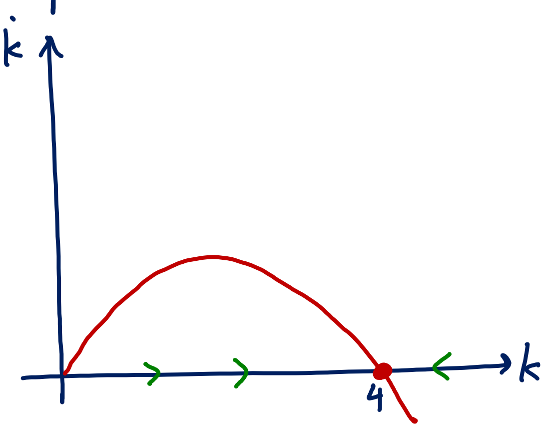
a. Plot the phase line of the model equation.
b. What are the equilibrium points? Are they dynamically stable or unstable?
c. What can you say about the capital-to-labor ratio in the long run?

a. Start by plotting the 2 terms $2k^{1/2}$ and k separately:



$2k^{\frac{1}{2}} = k$ when $k = 4$

Therefore, the phase line is:



b. $k = 4$ is a dynamically stable equilibrium

c. In the long run, $k \to 4$.

Problem 10. Consider the following model of a market with a single product in discrete time. The model variables are

P_t = unit price in period t $\quad$ Q_{dt} = quantity demanded in period t $\quad$ Q_{st} = quantity supplied in period t

The model equations are

$$Q_{dt} = Q_{st}$$
$$Q_{dt} = 5 - 2P_t$$
$$Q_{st} = -1 + 4P_{t-1}$$

a. Combine these equations to find a single equivalent difference equation.
b. Solve this difference equation.
c. What can you say about the price of the product in the long run?

a. $5 - 2P_t = -1 + 4P_{t-1}$

$\Leftrightarrow 2P_t + 4P_{t-1} = 6$

$\Leftrightarrow P_t + 2P_{t-1} = 3$

$\Leftrightarrow P_{t+1} + 2P_t = 3$ $\quad$ Here, $a = 2$, $c = 3 \Rightarrow \frac{c}{1+a} = 1$

b. $P_t = (P_0 - 1)(-2)^t + 1$

c. Since $b = -a = -2$, P_t is oscillatory and divergent.
In the long run, P_t does not converge to a constant.