Problem 1. Find the local optima of $f(x_1, x_2) = 8x_1^3 - 12x_1x_2 + x_2^3$.

The gradient of f is

$$\nabla f(x_1, x_2) = \begin{bmatrix} 24x_1^2 - 12x_2 \\ -12x_1 + 3x_2^2 \end{bmatrix}$$

Solving the system of equations given by $\nabla f(x_1, x_2) = 0$, we find that there are two critical points: $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 2)$.

The Hessian matrix of f is

$$H(x_1, x_2) = \begin{bmatrix} 48x_1 & -12 \\ -12 & 6x_2 \end{bmatrix}$$

At the critical point $(x_1, x_2) = (0, 0)$, we have

$$H(0, 0) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix} \qquad |H_1| = 0 \quad |H_2| = -144$$

and so $(0, 0)$ is a saddle point of f.

At the critical point $(x_1, x_2) = (1, 2)$, we have

$$H(1, 2) = \begin{bmatrix} 48 & -12 \\ -12 & 12 \end{bmatrix} \qquad |H_1| = 48 \quad |H_2| = 432$$

and so $f(1, 2) = -8$ is a local minimum of f.

Problem 2. Find the local optima of $f(x_1, x_2, x_3) = -x_1^4 - 2x_2^2 - x_3^2 + 4x_1x_2 + 2x_3$.

The gradient of f is

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} -4x_1^3 + 4x_2 \\ 4x_1 - 4x_2 \\ -2x_3 + 2 \end{bmatrix}$$

Solving the system of equations given by $\nabla f(x_1, x_2) = 0$, we find that there are three critical points: $(x_1, x_2, x_3) = (-1, -1, 1)$, $(x_1, x_2, x_3) = (0, 0, 1)$, and $(x_1, x_2, x_3) = (1, 1, 1)$.

The Hessian matrix of f is

$$H(x_1, x_2, x_3) = \begin{bmatrix} -12x_1^2 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

At the critical point $(x_1, x_2, x_3) = (-1, -1, 1)$, we have

$$H(-1, -1, 1) = \begin{bmatrix} -12 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad |H_1| = -12 \quad |H_2| = 32 \quad |H_3| = -64$$

and so $f(-1, -1, 1) = 2$ is a local maximum.

At the critical point $(x_1, x_2, x_3) = (0, 0, 1)$, we have

$$H(0, 0, 1) = \begin{bmatrix} 0 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad |H_1| = 0 \quad |H_2| = -16 \quad |H_3| = 32$$

and so $(-1, -1, 1)$ is saddle point of f.

At the critical point $(x_1, x_2, x_3) = (1, 1, 1)$, we have

$$H(1, 1, 1) = \begin{bmatrix} -12 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad |H_1| = -12 \quad |H_2| = 32 \quad |H_3| = -64$$

and so $f(1, 1, 1) = 2$ is a local maximum.

Problem 3. Is $f(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + 2x_3^2 + x_1x_3 + x_2x_3 + 8$ strictly convex or strictly concave? Why? Suppose $f(0,0,0) = 8$ is a local minimum of f. Is $f(0,0,0) = 8$ also an absolute minimum? Why?

The gradient of f is

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 + x_3 \\ 8x_2 + x_3 \\ x_1 + x_2 + 4x_3 \end{bmatrix}$$

The Hessian matrix of f is

$$H(x_1, x_2, x_3) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 8 & 1 \\ 1 & 1 & 4 \end{bmatrix} \quad \text{and so} \quad |H_1| = 2 \quad |H_2| = 16 \quad |H_3| = 54$$

Therefore, f is strictly convex. As a result, $f(0,0,0) = 8$ is an absolute minimum, since for a strictly convex function, every local minimum is also an absolute minimum.

Problem 4. Find the local optima of $f(x_1, x_2) = -x_1^2 + x_2^2$ subject to the constraint $x_1^2 + 4x_2^2 = 4$.

The Lagrangian function is

$$Z(x_1, x_2, \lambda) = -x_1^2 + x_2^2 + \lambda[4 - x_1^2 - 4x_2^2]$$

The gradient of Z is

$$\nabla Z(x_1, x_2, \lambda) = \begin{bmatrix} -2x_1 - 2x_1\lambda \\ 2x_2 - 8x_2\lambda \\ 4 - x_1^2 - 4x_2^2 \end{bmatrix}$$

Solving the system of equations given by $\nabla Z(x_1, x_2, \lambda) = 0$, we find that there are four critical points: $(x_1, x_2, \lambda) = (-2, 0, -1)$, $(x_1, x_2, \lambda) = (2, 0, -1)$, $(x_1, x_2, \lambda) = (0, -1, 1/4)$, $(x_1, x_2, \lambda) = (0, 1, 1/4)$.

The bordered Hessian matrix is

$$\overline{H}(x_1, x_2, \lambda) = \begin{bmatrix} 0 & 2x_1 & 8x_2 \\ 2x_1 & -2 - 2\lambda & 0 \\ 8x_2 & 0 & 2 - 8\lambda \end{bmatrix}$$

At the critical point $(x_1, x_2, \lambda) = (-2, 0, -1)$, we have

$$\overline{H}(-2, 0, -1) = \begin{bmatrix} 0 & -4 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} \qquad |\overline{H}_2| = -160$$

Therefore, $f(-2, 0) = -4$ is a local minimum.

At the critical point $(x_1, x_2, \lambda) = (2, 0, -1)$, we have

$$\overline{H}(2, 0, -1) = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix} \qquad |\overline{H}_2| = -160$$

Therefore, $f(2, 0) = -4$ is a local minimum.

At the critical point $(x_1, x_2, \lambda) = (0, -1, 1/4)$, we have

$$\overline{H}(0, -1, 1/4) = \begin{bmatrix} 0 & 0 & -8 \\ 0 & -5/2 & 0 \\ -8 & 0 & 0 \end{bmatrix} \qquad |\overline{H}_2| = 160$$

Therefore, $f(0, -1) = 1$ is a local maximum.

At the critical point $(x_1, x_2, \lambda) = (0, 1, 1/4)$, we have

$$\overline{H}(0, 1, 1/4) = \begin{bmatrix} 0 & 0 & 8 \\ 0 & -5/2 & 0 \\ 8 & 0 & 0 \end{bmatrix} \qquad |\overline{H}_2| = 160$$

Therefore, $f(0, 1) = 1$ is a local maximum.

Problem 5. Find the local optima of $f(x_1, x_2, x_3) = x_1x_2x_3$ subject to the constraint $x_1 + 2x_2 + 3x_3 = 6$.

The Lagrangian function is

$$Z(x_1, x_2, x_3, \lambda) = x_1x_2x_3 + \lambda[6 - x_1 - 2x_2 - 3x_3]$$

The gradient of Z is

$$\nabla Z(x_1, x_2, x_3, \lambda) = \begin{bmatrix} x_2x_3 - \lambda \\ x_1x_3 - 2\lambda \\ x_1x_2 - 3\lambda \\ 6 - x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

Solving the system of equations given by $\nabla Z(x_1, x_2, x_3, \lambda) = 0$, we find that there are four critical points: $(x_1, x_2, x_3, \lambda) = (0, 0, 2, 0)$, $(x_1, x_2, x_3, \lambda) = (0, 3, 0, 0)$, $(x_1, x_2, x_3, \lambda) = (6, 0, 0, 0)$, $(x_1, x_2, x_3, \lambda) = (2, 1, 2/3, 2/3)$.

The bordered Hessian matrix is

$$\overline{H}(x_1, x_2, x_3, \lambda) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & x_3 & x_2 \\ 2 & x_3 & 0 & x_1 \\ 3 & x_2 & x_1 & 0 \end{bmatrix}$$

At the critical point $(x_1, x_2, x_3, \lambda) = (0, 0, 2, 0)$, we have

$$\overline{H}(0, 0, 2, 0) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \qquad |\overline{H}_2| = 8 \quad |\overline{H}_3| = 36$$

Therefore, there is neither a local minimum nor a local maximum at $(x_1, x_2, x_3) = (0, 0, 2)$.

At the critical point $(x_1, x_2, x_3, \lambda) = (0, 3, 0, 0)$, we have

$$\overline{H}(0, 3, 0, 0) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \qquad |\overline{H}_2| = 0 \quad |\overline{H}_3| = 36$$

Therefore, there is neither a local minimum nor a local maximum at $(x_1, x_2, x_3) = (0, 3, 0)$.

At the critical point $(x_1, x_2, x_3, \lambda) = (6, 0, 0, 0)$, we have

$$\overline{H}(6, 0, 0, 0) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \\ 3 & 3 & 6 & 0 \end{bmatrix} \qquad |\overline{H}_2| = 0 \quad |\overline{H}_3| = 36$$

Therefore, there is neither a local minimum nor a local maximum at $(x_1, x_2, x_3) = (6, 0, 0)$.

At the critical point $(x_1, x_2, x_3, \lambda) = (2, 1, 2/3, 2/3)$, we have

$$\overline{H}(2, 1, 2/3, 2/3) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2/3 & 1 \\ 2 & 2/3 & 0 & 2 \\ 3 & 1 & 2 & 0 \end{bmatrix} \qquad |\overline{H}_2| = 8/3 \quad |\overline{H}_3| = -12$$

Therefore, $f(2, 1, 2/3) = 4/3$ is a local maximum.

Problem 6. Find the local optima of $f(x_1, x_2, x_3) = x_1$ subject to the constraints $x_1 - 4x_2^{1/2} - 8x_3^{1/2} = 0$ and $2x_2 + 4x_3 = 6$.

The Lagrangian function is

$$Z(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1 + \lambda_1(-x_1 + 4x_2^{1/2} + 8x_3^{1/2}) + \lambda_2(6 - 2x_2 - 4x_3)$$

The gradient of Z is

$$\nabla Z(x_1, x_2, x_3, \lambda_1, \lambda_2) = \begin{bmatrix} 1 - \lambda_1 \\ 2x_2^{-1/2}\lambda_1 - 2\lambda_2 \\ 4x_3^{-1/2}\lambda_1 - 4\lambda_2 \\ -x_1 + 4x_2^{1/2} + 8x_3^{1/2} \\ 6 - 2x_2 - 4x_3 \end{bmatrix}$$

Solving the system of equations given by $\nabla Z(x_1, x_2, x_3, \lambda_1, \lambda_2) = 0$, we find that there is one critical point: $(x_1, x_2, x_3, \lambda_1, \lambda_2) = (12, 1, 1, 1, 1)$.

The bordered Hessian matrix is

$$\overline{H}(x_1, x_2, x_3, \lambda_1, \lambda_2) = \begin{bmatrix} 0 & 0 & 1 & -2x_2^{-1/2} & -4x_3^{-1/2} \\ 0 & 0 & 0 & 2 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ -2x_2^{-1/2} & 2 & 0 & -x_2^{-3/2}\lambda_1 & 0 \\ -4x_3^{-1/2} & 4 & 0 & 0 & -2x_3^{-3/2}\lambda_1 \end{bmatrix}$$

At the critical point $(x_1, x_2, x_3, \lambda_1, \lambda_2) = (12, 1, 1, 1, 1)$, we have

$$\overline{H}(12, 1, 1, 1, 1) = \begin{bmatrix} 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 2 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & -1 & 0 \\ -4 & 4 & 0 & 0 & -2 \end{bmatrix} \qquad |\overline{H}_3| = -24$$

Therefore, $f(12, 1, 1) = 12$ is a local maximum.

Problem 7. Consider a firm that produces and sells two products. Below is a model that represents the firm's profit maximization problem.

- Variables:

$$Q_1 = \text{quantity of product 1 produced and sold} \qquad P_1 = \text{unit price of product 1}$$
$$Q_2 = \text{quantity of product 2 produced and sold} \qquad P_2 = \text{unit price of product 2}$$

- Model:

$$\begin{aligned} \text{maximize} \quad & P_1Q_1 + P_2Q_2 - 12 - 4Q_1 - 8Q_2 \\ \text{subject to} \quad & P_1 = 46 - 3Q_1 \\ & P_2 = 32 - 2Q_2 \end{aligned}$$

a. Describe the objective function and the constraints of the model. (e.g. What are the prices of the products? What are the costs of production? Do the prices depend on demand?)

b. By substituting the constraints into the objective function, find the local maximum values of profit.

c. Using the Lagrange multiplier method, find the local maximum values of profit.

d. What do your answers from parts b and c tell you about what the firm should do?

a. In this model, prices depend on the quantity produced and sold, as given in the two equality constraints. There is a fixed cost of 12; each unit of quantity 1 costs 4, while each unit of quantity costs 8.

b. Let π be the profit function. By substitution, we have that

$$\pi(Q_1, Q_2) = 42Q_1 - 3Q_1^2 + 24Q_2 - 2Q_2^2 - 12$$

The gradient of π is

$$\nabla\pi(Q_1, Q_2) = \begin{bmatrix} 42 - 6Q_1 \\ 24 - 4Q_2 \end{bmatrix}$$

Solving the system of equations given by $\nabla\pi(Q_1, Q_2) = 0$, we find that there is one critical point: $(Q_1, Q_2) = (7, 6)$. The Hessian of π is

$$H(Q_1, Q_2) = \begin{bmatrix} -6 & 0 \\ 0 & -4 \end{bmatrix}$$

At the critical point $(Q_1, Q_2) = (7, 6)$, we have

$$H(7,6) = \begin{bmatrix} -6 & 0 \\ 0 & -4 \end{bmatrix} \qquad |H_1| = -6 \quad |H_2| = 24$$

Therefore $\pi(7, 6) = 207$ is a local maximum.

c. Let $f(Q_1, Q_2, P_1, P_2) = P_1Q_1 + P_2Q_2 - 12 - 4Q_1 - 8Q_2$, $g_1(Q_1, Q_2, P_1, P_2) = P_1 + 3Q_1$, $c_1 = 46$, $g_2(Q_1, Q_2, P_1, P_2) = P_2 + 2Q_2$, $c_2 = 32$. The Lagrangian function is

$$Z(Q_1, Q_2, P_1, P_2, \lambda_1, \lambda_2) = P_1Q_1 + P_2Q_2 - 12 - 4Q_1 - 8Q_2 + \lambda_1(46 - P_1 - 3Q_1) + \lambda_2(32 - P_2 - 2Q_2)$$

The gradient of Z is

$$\nabla Z(Q_1, Q_2, P_1, P_2, \lambda_1, \lambda_2) = \begin{bmatrix} P_1 - 4 - 3\lambda_1 \\ P_2 - 8 - 2\lambda_2 \\ Q_1 - \lambda_1 \\ Q_2 - \lambda_2 \\ 46 - P_1 - 3Q_1 \\ 32 - P_2 - 2Q_2 \end{bmatrix}$$

Solving the system of equations given by $\nabla Z(Q_1, Q_2, P_1, P_2, \lambda_1, \lambda_2) = 0$, we find that there is one critical point: $(Q_1, Q_2, P_1, P_2, \lambda_1, \lambda_2) = (7, 6, 25, 20, 7, 6)$.

The bordered Hessian is

$$\overline{H}(Q_1, Q_2, P_1, P_2, \lambda_1, \lambda_2) = \begin{bmatrix} 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

At the critical point $(Q_1, Q_2, P_1, P_2, \lambda_1, \lambda_2) = (7, 6, 25, 20, 7, 6)$, we have

$$\overline{H}(7, 6, 25, 20, 7, 6) = \begin{bmatrix} 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad |\overline{H}_3| = -24 \quad |\overline{H}_4| = 24$$

Therefore, $f(7, 6, 25, 20) = 207$ is a local maximum.

d. Assuming a local maximum is sufficient, the firm should set the price of product 1 to 25 and the price of product 2 to 20, resulting in 7 units of product 1 and 6 units of product 2 sold.

Problem 8. Consider a firm that produces a good that requires two inputs to produce. In particular, x_1 units of input 1 and x_2 units of input 2 yield $3x_1^{1/3}x_2^{1/3}$ units of the good. Each unit of input 1 costs \$20, and each unit of input 2 costs \$160.

a. Using the variables x_1 and x_2 defined above, write cost as a function of x_1 and x_2: $c(x_1, x_2) = \ldots$
b. The firm needs to produce 24 units of the product. Using the variables x_1 and x_2 defined above, write an equality constraint that models this.
c. Using the Lagrange multiplier method, find the local minimum values of cost.
d. What does your answer from part c tell you about what the firm should do?

a. $c(x_1, x_2) = 20x_1 + 160x_2$

b. $3x_1^{1/3}x_2^{1/3} = 24$

c. The Lagrangian function is

$$Z(x_1, x_2, \lambda) = 20x_1 + 160x_2 + \lambda(24 - 3x_1^{1/3}x_2^{1/3})$$

The gradient of Z is

$$\nabla Z(x_1, x_2, \lambda) = \begin{bmatrix} 20 - \lambda x_1^{-2/3}x_2^{1/3} \\ 160 - \lambda x_1^{1/3}x_2^{-2/3} \\ 24 - 3x_1^{1/3}x_2^{1/3} \end{bmatrix}$$

Solving the system of equations given by $\nabla Z(x_1, x_2, \lambda) = 0$, we find that there is one critical point: $(x_1, x_2, \lambda) = (64, 8, 160)$.

The bordered Hessian is

$$\overline{H}(x_1, x_2, \lambda) = \begin{bmatrix} 0 & x_1^{-2/3}x_2^{1/3} & x_1^{1/3}x_2^{-2/3} \\ x_1^{-2/3}x_2^{1/3} & \frac{2}{3}x_1^{-5/3}x_2^{1/3}\lambda & -\frac{1}{3}x_1^{-2/3}x_2^{-2/3}\lambda \\ x_1^{1/3}x_2^{-2/3} & -\frac{1}{3}x_1^{-2/3}x_2^{-2/3}\lambda & \frac{2}{3}x_1^{1/3}x_2^{-5/3}\lambda \end{bmatrix}$$

At the critical point $(x_1, x_2, \lambda) = (64, 8, 160)$, we have

$$\overline{H}(64, 8, 160) = \begin{bmatrix} 0 & 1/8 & 1 \\ 1/8 & 5/24 & -5/6 \\ 1 & -5/6 & 40/3 \end{bmatrix} \qquad |\overline{H}_3| = -\frac{5}{8}$$

Therefore, $c(64, 8) = 2560$ is a local minimum.

d. Assuming a local minimum is sufficient, the firm should use 64 units of input 1 and 8 units of input 2.