

Name:

SM286A – Mathematics for Economics
Asst. Prof. Nelson Uhan

Fall 2015

Exam 3

Instructions

- You have 50 minutes to complete this exam.
- There are 7 problems on this exam, worth a total of 100 points.
- You may consult one 3 in \times 5 in index card of handwritten notes. You may not consult any other outside materials.
- You may use a calculator.
- **Show all your work.** Your answers should be legible and clearly labeled. It is your responsibility to make sure that I understand what you are doing. You will be awarded partial credit if your work merits it.
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until the end of 4th period.**

Score

Problem	Maximum Points	Your Points
1	15	
2	15	
3	10	
4	15	
5	15	
6	15	
7	15	
Total	100	

Problem 1. (15 total points) Find the critical points of

$$f(x_1, x_2, x_3) = x_1^2 + x_1x_3 + x_2^3 - 12x_2 + 3x_3^2$$

Do not determine whether there is a local minimum, local maximum, or saddle point at each critical point.

Problem 2. (15 total points) Consider the function $f(x_1, x_2, x_3) = 2x_1^2 + x_2^4 - 4x_2 + x_3^2$. The gradient of f is

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 \\ 4x_2^3 - 4 \\ 2x_3 \end{bmatrix}$$

The single critical point of f is $(x_1, x_2, x_3) = (0, 1, 0)$. Determine whether there is a local minimum, local maximum, or saddle point at this critical point.

Problem 3. (10 total points)

a. (5 points) Let $f(x_1, x_2, x_3) = -2x_1^2 - x_2^2 - x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$. The Hessian of f is

$$H = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Is f strictly convex or strictly concave? Why?

b. (5 points) The function

$$f(x_1, x_2, x_3) = 4x_1^2 + 3x_1x_2 + x_1x_3 + x_2^2 + 2x_2x_3 + 8x_3^2 + 25$$

is strictly convex, and $f(0, 0, 0) = 25$ is a local minimum. Is $f(0, 0, 0) = 25$ also an absolute minimum? Why?

Problem 4. (15 total points) Using the Lagrange multiplier method, find the critical points of

$$f(x_1, x_2, x_3) = x_1^3 - 3x_1 + x_3^2 \quad \text{subject to the equality constraint} \quad 3x_1 + 2x_2 = 9$$

Do not determine whether there is a local minimum or local maximum at each critical point.

Problem 5. (15 total points) Consider finding the local optima of

$$f(x_1, x_2, x_3) = 4x_2 - x_2^4 \quad \text{subject to the equality constraints} \quad 3x_1 + 2x_2 = 5 \quad \text{and} \quad x_2 + 3x_3 = 4$$

using the Lagrange multiplier method. The gradient of the Lagrangian function is

$$\nabla Z(x_1, x_2, x_3, \lambda_1, \lambda_2) = \begin{bmatrix} -3\lambda_1 \\ 4 - 4x_2^3 - 2\lambda_1 - \lambda_2 \\ -3\lambda_2 \\ 5 - 3x_1 - 2x_2 \\ 4 - x_2 - 3x_3 \end{bmatrix}$$

The single critical point is $(x_1, x_2, x_3, \lambda_1, \lambda_2) = (1, 1, 1, 0, 0)$. Determine whether there is a local minimum or local maximum at this critical point.

Problem 6. (15 total points) Consider a firm that produces and sells two products. Below is a model that represents the firm's profit maximization problem.

- Variables:

Q_1 = quantity of product 1 produced and sold

P_1 = unit price of product 1

Q_2 = quantity of product 2 produced and sold

P_2 = unit price of product 2

- Model:

$$\text{maximize } P_1Q_1 + P_2Q_2 - 3Q_1 - 2Q_2 - 100$$

$$\text{subject to } P_1 = 63 - Q_1$$

$$P_2 = 82 - 5Q_2$$

- a. (10 points) Find the local maximum values of profit using the method of your choice. You must show why the values you found are local maximum values.

- b. (5 points) What does your answer from part a tell you about what the firm should do?

Problem 7. (15 total points) Suppose that you are interested in dividing your savings between two mutual funds with expected returns of 8% and 12% respectively. You want to maximize the total expected return of your portfolio while keeping risk at a fixed level. To measure risk, you use the variance of the return of your portfolio: when a fraction x_1 of your savings is invested in Fund 1 and x_2 in Fund 2, the variance of the return is $100x_1^2 + 200x_2^2 + 50x_1x_2$.

a. (5 points) Using the variables defined above, write the total expected return r as a function of x_1 and x_2 : i.e. $r(x_1, x_2) = \dots$

b. (5 points) You want to keep the variance of the return of your portfolio equal to 82. Using the variables defined above, write an equality constraint that models this.

c. (5 points) Why does the equality constraint $x_1 + x_2 = 1$ make sense for this problem?