Lesson 2. Introduction to Matrices and Vectors

1 Overview

- Last time: models with three variables and three equations
- What if we have a model with hundreds of variables and equations? Thousands?
- **Matrix algebra** enables us to handle large systems of linear equations in a concise way
  - Important for equilibrium analysis (a.k.a. comparative statics), econometrics, optimization
  - Some types of nonlinear systems can be transformed into or approximated by systems of linear equations

2 What is a matrix?

- A **matrix** is a rectangular array of numbers, symbols, or expressions
- For example:

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad c = \begin{bmatrix} 22 & 12 & 10 \end{bmatrix}$$

- The individual items in a matrix are called its **elements** (or **entries**)
- By convention:

$$\begin{aligned} a_{ij} &= \text{the element in the } i\text{th row and } j\text{th column of matrix } A \\ &= \text{"the } ij \text{ element of } A\text{"} \end{aligned}$$

- The **dimension** (or **size**) of a matrix with m rows and n columns is $m \times n$ ("m by n")
- "Row then column!"

Example 1.

a. What is the dimension of A? x? c?
b. What is a_{23}? a_{32}? c_{12}?

- A **row vector** is a matrix with only one row
- A **column vector** is a matrix with only one column
- For notation purposes, we denote a row vector with a prime symbol – for example:

- This notation actually means a little more – we'll come back to this in a later lesson

3 Matrix equality, addition and subtraction

- Two matrices are equal if and only if
  - they have the same dimension
  - their corresponding elements are identical
    - i.e. the *ij* element of one matrix is equal to the *ij* element of the other
- For example:

$$\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

- When we add two matrices of the same dimension, we get another matrix of the same dimension
- We add two matrices by adding their corresponding elements
- We subtract two matrices by subtracting their corresponding elements

Example 2. Compute the following.

a. $\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$ b. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

- Note: you cannot add or subtract two matrices of different dimension!

4 Scalar multiplication

- When we multiply a matrix by a scalar (a number), we get another matrix of the same dimension
- We multiply a matrix by a scalar by multiplying each element of the matrix by the scalar

Example 3. Find the following products.

$$\text{a.} \quad 7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} \qquad\qquad \text{b.} \quad -1\begin{bmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \end{bmatrix}$$

5 Multiplying vectors: the inner product

- The **inner product** (or **dot product**) of vectors $u = [u_1, u_2, \dots, u_n]$ and $v = [v_1, v_2, \dots, v_n]$ is
- The inner product is well-defined only when u and v have the same number of elements
- u and v can be row or column vectors

6 Matrix multiplication

- How do we multiply two matrices together?
- Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix
  - Note: (# columns of A) = (# rows of B)

Example 4. Quick check: What does the ith row of A look like? What does the jth column of B look like?

- The product AB is an $m \times p$ matrix

- To get the ij element of AB:
  - We take the ith row of A and jth column of B
  - Multiply corresponding their elements and add it all up:

- In other words, the ij element of AB is the inner product of the ith row of A and the jth column of B

Example 5. Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 2 \\ 9 & 1 \end{bmatrix}$$

a. What is the dimension of AB?
b. Find AB.

- Note: order of multiplication matters! Usually, $AB \neq BA$
- For instance, in Example 5, BA is not well-defined because (# columns of B) $\neq$ (# rows of A)

Example 6. Let

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a. What is the dimension of Ax?
b. Find Ax.

7 Systems of linear equations using matrices and vectors

Example 7. Write the following system of linear equations using matrices and vectors.

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

8 Review: the $\sum$ notation

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n \qquad (m \text{ and } n \text{ are integers, } m \leq n)$$

Example 8. Expand the following summations.

$$\text{a.} \quad \sum_{i=3}^{7} x_i \qquad \text{b.} \quad \sum_{i=0}^{4} a_i x^i \qquad \text{c.} \quad \sum_{i=3}^{6} \frac{1}{i}$$

- Consider the product AB of $m \times n$ matrix A and $n \times p$ matrix B
- Recall that the ij element of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i,n-1}b_{n-1,j} + a_{in}b_{nj}$$

- We can rewrite this using $\sum$ notation as