SM286A – Mathematics for Economics Fall 2015
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Lesson 3. Useful and Interesting Properties of Matrix Algebra

0 Warm up

Example 1. Let

$$A = \begin{bmatrix} 6 & -5 & 1 \\ 1 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix}$$

Find AB and BA. Do we have $AB = BA$?

Example 2. Let $u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $v' = \begin{bmatrix} 1 & 8 & 3 \end{bmatrix}$. Find uv'.

Example 3. Let $u' = \begin{bmatrix} 4 & 3 \end{bmatrix}$ and $v = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$. Find $u'v$.

1 Important points from the warm up

- Order of multiplication matters! Typically, $AB \neq BA$
- u is a $m \times 1$ column vector, v' is a $1 \times n$ row vector

  $\Rightarrow$ uv' has dimension

- u' is a $1 \times n$ row vector, v is a $n \times 1$ column vector

  $\Rightarrow$ $u'v$ has dimension

  - As as result, $u'v$ can be viewed as a scalar

2 Matrices act like scalars under addition

- $A - B = A + (-B)$
- **Commutative law.** For any two matrices A, B:
- **Associative law.** For any three matrices A, B, C:

3 Matrices don't always act like scalars under multiplication

- As we saw in Example 1, matrix multiplicative is not commutative: $AB \neq BA$
- Since order matters in multiplication, we have terminology that specifies the order
- In the product AB:
  - B is **premultiplied** by A
  - A is **postmultiplied** by B
- **Associative law.** For any three matrices A, B, C:

4 The distributive law

- **Distributive law.** For any three matrices A, B, C:

5 With your neighbor

Example 4. Let

$$A = \begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 0 \\ 2 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 6 & 2 \end{bmatrix}$$

Compute the following:

a. AC b. BC c. $(B + A)C$

Example 5. Compute the following:

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 3 & -2 & 4 \\ -9 & 8 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 6. Compute the following:

a. $\begin{bmatrix} -2 & 4 & 1 \\ 8 & 6 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$

Example 7. Let

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

Find CD and CE.

6 Identity matrices

- An **identity matrix** is a square matrix with 1s in its principal diagonal (northwest to southeast) and 0s everywhere else:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- I_n is the $n \times n$ identity matrix
- What happened in Example 5?
- The identity matrix plays the role that "1" has with scalars
- For any matrix A, we have
- We can insert or delete an identity matrix without affecting a matrix product:
- What is $(I_n)^2 = (I_n)(I_n)$? How about $(I_n)^k$ for any integer $k \geq 1$?

7 Null matrices

- A **null matrix** (or **zero matrix**) is a matrix whose elements are all 0
- A null matrix is <u>not</u> restricted to being square
  - It's important to keep track of a null matrix's dimension
- We denote a null matrix by 0:

$$\underset{(2\times 2)}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underset{(2\times 3)}{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- What happened in Example 6a?
- The null matrix plays the role that "0" has with scalars
- For any matrix A, we have:

8 Matrix algebra can be weird

- Unlike algebra with scalars, $AB = 0$ does <u>not</u> necessarily imply either $A = 0$ or $B = 0$
  - To illustrate, recall Example 6b
- Also unlike algebra with scalars, $CD = CE$ does <u>not</u> necessarily imply $D = E$
  - To illustrate, recall Example 7