SM286A – Mathematics for Economics Fall 2015
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# Lesson 4. Transposes and Inverses

## 1 The transpose of a matrix

- Let $A$ be an $m \times n$ matrix
- The **transpose** of $A$ is denoted by $A'$
    - $A'$ has dimension $n \times m$
    - The columns of $A$ are the rows of $A'$
    - (The rows of $A$ are the columns of $A'$)

**Example 1.** Let $A = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 9 & 7 \\ 3 & 7 & 1 \end{bmatrix}$. Find $A'$ and $B'$.

- A matrix $B$ is **symmetric** if $B = B'$
    - What are some examples of symmetric matrices?
- Properties of transposes:
    - $(A')' = A$
    - $(A + B)' = A' + B'$
    - $(AB)' = B'A'$

**Example 2.** Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 0 \\ 9 & 0 & 2 \end{bmatrix}$. Find $(AB)'$ and $B'A'$.

## 2 The inverse matrix

- Let's start with some motivation

**Example 3.** Solve for $x$: $ax = d$ ($a \neq 0$)

- Recall from the homework that we can write the partial market equilibrium model

$$\begin{aligned} Q_d &= Q_s \\ Q_d &= a - bP \\ Q_s &= -c + dP \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} Q_d - Q_s \quad\quad &= 0 \\ Q_d \quad\quad + bP &= a \\ Q_s - dP &= -c \end{aligned}$$

using matrices as

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}}_{d}$$

- It would be nice if we could write:

$$Ax = d \qquad \Leftrightarrow \qquad A^{-1}Ax = A^{-1}d \qquad \Leftrightarrow \qquad x = A^{-1}d$$

- Let $A$ be a $n \times n$ (square) matrix
- The **inverse** of a matrix $A$ is denoted by $A^{-1}$
  - $A^{-1}$ is also $n \times n$
  - $A^{-1}$ satisfies
- Properties of inverses:
  - $A^{-1}$ is defined only if $A$ is square
  - $A^{-1}$ does not necessarily exist
    - $A$ is **nonsingular** if it has an inverse (a.k.a. **invertible**)
    - $A$ is **singular** if it has no inverse
  - If $A^{-1}$ exists, then $A^{-1}$ is unique
  - $AA^{-1} = I$ implies $A^{-1}A = I$ and vice-versa

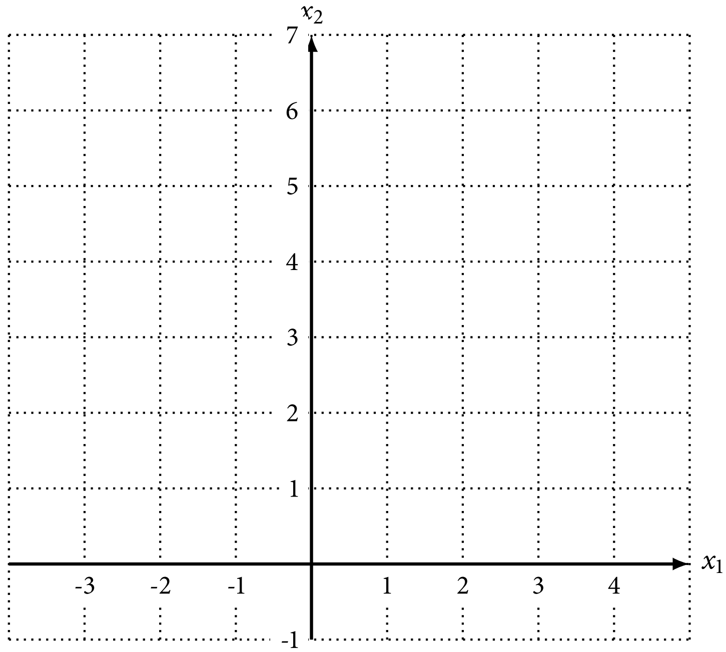
**Example 4.** Let $A = \begin{bmatrix} 1 & 0 \\ 12 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 1 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ -4 & 1/3 \end{bmatrix}$. Test whether any of these matrices is the inverse of another.

- More properties of inverses: ($A$ and $B$ are nonsingular and square)
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A')^{-1} = (A^{-1})'$

**Example 5.** Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

## 3 When is a matrix nonsingular? Linear dependence and independence

**Example 6.** Graph the vectors $v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.




- A **linear combination** is a sum of scalar multiples of vectors, e.g.

  for some vectors $v_1, \ldots, v_n$ and scalars $k_1, \ldots, k_n$

- A set of vectors $v_1, \ldots, v_n$ is **linearly dependent** if and only if any one of them can be expressed as a linear combination of the remaining vectors

- Otherwise, this set of vectors is **linearly independent**

- Special case: two vectors $v_1$ and $v_2$ are linearly dependent if and only if

**Example 7.** Are the vectors in Example 6 linearly dependent or linearly independent?

- Why do we care about linear dependence or independence?
- Consider the following system of linear equations

$$\begin{aligned} 10x_1 + 4x_2 &= 8 \\ 5x_1 + 2x_2 &= 2 \end{aligned}$$

- We can rewrite the above system using matrices as
- If $A$ were nonsingular (i.e. $A^{-1}$ exists), then we could find a solution:
- But does the above system have a solution? Why?
- The lines have the same slope because
- Equivalently,

- In general: a square matrix is nonsingular if and only if its rows are linearly independent
  - Also: a square matrix is nonsingular if and only if its columns are linearly independent

**Example 8.** Show that $A = \begin{bmatrix} 4 & 3 & 5 \\ 1 & 0 & 2 \\ 8 & 6 & 10 \end{bmatrix}$ is singular.

**Example 9.** Show that $B = \begin{bmatrix} 0 & 2 & -1 \\ -3 & -9 & 3 \\ 7 & 5 & 1 \end{bmatrix}$ is singular.

## 4 Next lesson...

- A systematic way of determining linear independence/dependence, nonsingularity, and solutions to systems of linear equations