Lesson 5. Reduced Row Echelon Form

0 Warm up

Example 1. Rewrite the system of linear equations

$$\begin{aligned} 2x + 4z &= 2 - 8y \\ z &= 5 - 2x - 5y \\ 4x + 10y - z &= 1 \end{aligned} \tag{A}$$

using matrices, assuming the three variables are arranged in the order x, y, z.

1 The augmented matrix

- The matrix $\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}$ is the **coefficient matrix** for the system (A)
  - Columns of the coefficient matrix $\Leftrightarrow$ Variables in the system
- The matrix $\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$ is the **augmented matrix** for the system (A)
- To solve (A), we will work with its augmented matrix

2 Elementary row operations

- The **elementary row operations** are
  1. Divide a row by a nonzero number
  2. Subtract a multiple of a row from another row
  3. Interchange two rows
- Performing an elementary row operation on an augmented matrix
  - $\Rightarrow$ Get new augmented matrix representing new system of equations AND <u>new system of equations has the same solutions as the original</u>

3 Reduced row echelon form

- How should we use these elementary row operations?
- A matrix is in **row echelon form** if
  1. Nonzero rows appear above the zero rows
  2. In any nonzero row, the first nonzero entry is a 1 (this is called the **leading one**)
  3. The leading one in a nonzero row appears to the left of the leading one in any lower row
- A matrix is in **reduced row echelon form (RREF)** if conditions 1-3 above are satisfied and
  4. If a column contains a leading one, then all the other entries in that column are 0

Example 2. Are the following matrices in row echelon form? RREF?

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

- Why is RREF useful?
- Take a look at the 4th matrix in Example 2
- Suppose it is the augmented matrix of a system of equations with variables x, y, z
- The system is

⇒ If we can transform augmented matrices into RREF in a systematic way, then we can solve systems of linear equations in a systematic way

- Strategy:
  - First, transform entries in the lower left into 0s ⇒ row echelon form
  - Second, transform entries above the leading ones into 0s ⇒ RREF

Example 3. Transform the augmented matrix $\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$ of system (A) into RREF.

- Suppose an augmented matrix is transformed into RREF
- A column that contains a leading one is called a **leading column**
- A variable that corresponds to a leading column is called a **leading variable**
- The non-leading variables are called **free variables**
- To find all solutions to the system, just solve the equations for the leading variables in terms of the free variables

Example 4. In the RREF matrix you found in Example 3, what are the leading variables? What are the free variables? Use the RREF matrix to solve the system (A).

Example 5. Suppose the RREF of the augmented matrix of a system of linear equations is

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the corresponding system? What are the leading variables? What are the free variables? What are the solutions of this system?

Example 6. Suppose the RREF of the augmented matrix of a system of linear equations is

$$\begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the corresponding system? What are the leading variables? What are the free variables? What are the solutions of this system?

4 If we have time...

Example 7 (Also a homework problem)**.** Solve the following system of equations by finding the RREF of the corresponding augmented matrix.

$$\begin{aligned} x + 2y + 3z &= 4 \\ 3x + 4y + z &= 5 \\ 2x + y + 3z &= 6 \end{aligned}$$