Lesson 6. Reduced Row Echelon Form, cont.

0 Warm up

Example 1. Consider the following system of equations:

$$\begin{aligned} x + y - 2z &= 1 \\ -x + 10z &= -1 \\ 2x + 3y + 4z &= 2 \end{aligned}$$

a. Form the augmented matrix for this system.
b. Solve this system by putting its augmented matrix into RREF.

1 Last time

- Elementary row operations and RREF
- To solve a system of linear equations, we
  1. Form its augmented matrix
  2. Find the RREF of the augmented matrix
  3. Solve for the leading variables in terms of the free variables
- Three possible outcomes:
  1. The system has a unique solution
  2. The system has infinitely many solutions (if there is at least one free variable)
  3. The system is inconsistent (if the RREF implies $0 = 1$)
- What else can RREF tell us?

2 The rank of a matrix

- Let $\mathrm{rref}(A)$ denote the reduced row echelon form of matrix A
- The **rank** of matrix A – denoted by $r(A)$ – is the number of leading 1s in $\mathrm{rref}(A)$

Example 2. Let $A = \begin{bmatrix} 0 & -11 & -4 \\ 2 & 6 & 2 \\ 4 & 1 & 0 \end{bmatrix}$. We have that $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1/11 \\ 0 & 1 & 4/11 \\ 0 & 0 & 0 \end{bmatrix}$. What is $r(A)$?

Example 3. What is the maximum possible rank of an $m \times n$ matrix?

- $r(A)$ is the maximum number of linearly independent <u>rows</u> that can be found in A
- Similarly, $r(A)$ is the maximum number of linearly independent <u>columns</u> that can be found in A
- Recall that a square matrix A is nonsingular if and only if its rows/columns are linearly independent

Example 4. Is A in Example 2 nonsingular? Why?

$\Rightarrow$ An $n \times n$ matrix is nonsingular if and only if its rank is

3 Finding the inverse of a matrix

- To find the inverse of an $n \times n$ matrix A, compute $\text{rref}\left(\begin{bmatrix} A & I_n \end{bmatrix}\right)$
  - If $\text{rref}\left(\begin{bmatrix} A & I_n \end{bmatrix}\right)$ has the form $\begin{bmatrix} I_n & B \end{bmatrix}$, then A is invertible and $A^{-1} = B$
  - If $\text{rref}\left(\begin{bmatrix} A & I_n \end{bmatrix}\right)$ has another form (i.e., the left side fails to be I_n), then A is not invertible

Example 5. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

4 A two-commodity market equilibrium model

- Suppose we have a market with two commodities
- The unit prices of the two commodities are P_1 and P_2
- The quantities demanded – Q_{d1} and Q_{d2} – and the quantities supplied – Q_{s1} and Q_{s2} – of the two commodities are given by

$$Q_{d1} = 70 - 2P_1 + P_2 \qquad Q_{s1} = -14 + 3P_1$$
$$Q_{d2} = 105 + P_1 - P_2 \qquad Q_{s2} = -7 + 2P_2$$

- What is the relationship between the two commodities?

- Equilibrium condition – demand equals supply for each commodity:

$$Q_{d1} = Q_{s1} \qquad Q_{d2} = Q_{s2}$$

- Using the equilibrium condition, we can simplify the two-commodity market equilibrium model:

- Equivalently, in matrix form:

Example 6. Solve for the equilibrium prices of this two-commodity market equilibrium model by finding the RREF of the augmented matrix of the above system.

Example 7. Solve for the equilibrium prices of this two-commodity market equilibrium model by finding the inverse of the coefficient matrix of the above system.

5 If we have time...

Example 8 (Also a homework problem)**.** What is the rank of $\begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 8 & 0 & 0 & 8 \end{bmatrix}$? Is this matrix nonsingular?