Lesson 9. Leontief Input-Output Models

0 Warm up

Example 1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Verify that $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

1 The setup

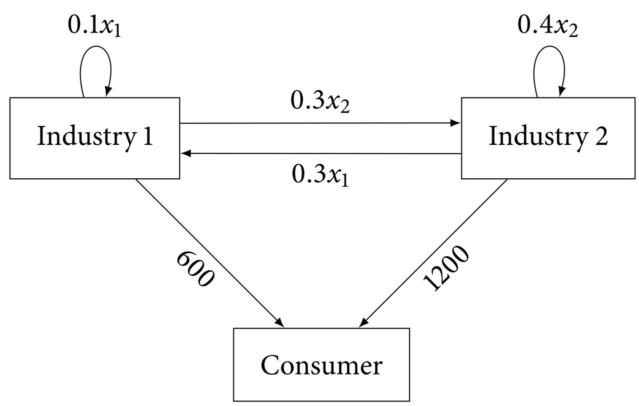
- Economy with a number of industries
- Each industry produces a single homogeneous product
- **Input demand** for each product:
    - Outputs for one industry are used as input for another industry
- **Final demand** for each product:
    - e.g. consumer households, government sector, foreign countries
- What output should each industry produce to satisfy the total demand for all products?

2 An example

- Economy with 2 industries/products
- Let

$$x_1 = \text{output of industry 1, in dollars}$$
$$x_2 = \text{output of industry 2, in dollars}$$

- Required inputs and demand, in dollars:



- Output of each industry must be just sufficient to meet the required inputs and demand:

- We can rewrite this system of equations in matrix form:

- Using the inverse of the coefficient matrix, we can find the required output levels x_1 and x_2:

- Note that \$1 of product 1 requires \$0.1 of product 1 and \$0.3 of product 2

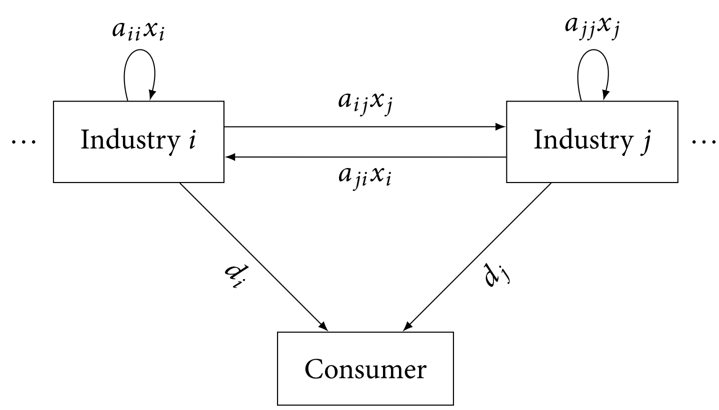
⇒ \$1 of product 1 requires \$(1 – 0.1 – 0.3) = \$0.6 of **primary inputs**

  - Inputs besides those supplied by the industries in the economy, e.g. labor

- In addition, note that \$1 of product 2 requires \$0.3 of product 1 and \$0.4 of product 2
- Therefore, \$1 of product 2 requires [ ] of primary inputs
- What is the total required amount of primary inputs for this economy?

3 Generalization to economies with n industries

- Let x_i = output of industry i, in dollars $(i = 1, \ldots, n)$
- Required inputs and demand, in dollars:



- a_{ij} = dollars of product i required to produce one dollar of product j $(i = 1, \ldots, n; j = 1, \ldots, n)$
- Output of each industry must be just sufficient to meet the required inputs and demand:

- We can rewrite this system of equations in matrix form:

- Let A be the matrix of a_{ij} values – the **input matrix**
- We can represent this system as

- $I - A$ is the **Leontief matrix**
- If $I - A$ is nonsingular, then we can solve for the required outputs:

- \$1 of product j requires a_{1j} of product 1, a_{2j} of product 2, …, a_{nj} of product n

⇒ Let a_{0j} represent the primary inputs required for \$1 of product j:

- The total required amount of primary inputs for this economy is:

Example 2. Consider an economy with 3 industries/products. Suppose the input matrix is

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$$

a. What are the primary inputs of each product?
b. Assume the final demands for the 3 products are d_1, d_2 and d_3, respectively. What is the required output of the 3 industries?
c. If $d_1 = 10$, $d_2 = 5$, and $d_3 = 6$, find the total required amount of primary inputs for this economy.

4 The existence of nonnegative solutions

- Leontief matrix $I - A$ is nonsingular $\Rightarrow$ solutions to the model exist
- Ideally, the solutions to the model would also be nonnegative
- When does this happen?
- The Leontief matrix $I - A$ has the form:

$$\begin{bmatrix} 1-a_{11} & -a_{12} & -a_{13} & \dots & -a_{1n} \\ -a_{21} & 1-a_{22} & -a_{23} & \dots & -a_{2n} \\ -a_{31} & -a_{32} & 1-a_{33} & \dots & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \dots & 1-a_{nn} \end{bmatrix}$$

- The **leading principal minors** of $I - A$ are:

- **Hawkins-Simon condition.** The system $(I - A)x = d$ has a nonnegative solution if and only if the leading principal minors of $I - A$ are all positive

Example 3. Verify the Hawkins-Simon condition holds in Example 2.