# Lesson 10. First-Order Linear Differential Equations with Constant Coefficients

## 0 Warm up

**Example 1.** Suppose the price $P$ of a product is a function of time $t$: $P(t) = Ae^{-kt} + B$, with $k > 0$.

a. What is the price at time $t = 0$?
b. What is the price as $t \to \infty$?
c. What is $dP/dt$?
d. What information does $dP/dt$ tell you?

## 1 Overview

- In economics, we often want to study how quantities change over time
    - e.g., price of a product, population growth, unemployment rate
- The **dynamics** of these quantities can be modeled using **differential equations**

## 2 First-order linear differential equations

$$\text{Given} \quad \frac{dy}{dt} + u(t)y = w(t), \quad \text{what is} \quad y(t)?$$

- Note: the solution to a differential equation is a function
- In this lesson, we will consider constant coefficients: $u(t) = a$, $w(t) = b$

## 3 The homogeneous case

- What happens when $u(t) = a$ and $w(t) = 0$? We want to find $y(t)$, given

$$\frac{dy}{dt} + ay = 0$$

- We can solve for $y(t)$:

- The **general solution** is

- How do we find $A$? Note that

- The **definite solution** is

## 4 The nonhomogeneous case

- Now what happens when $u(t) = a \neq 0$ and $w(t) = b$? We want to find $y(t)$, given

$$\frac{dy}{dt} + ay = b$$

- Let's start by multiplying both sides by $e^{at}$

$$e^{at}\frac{dy}{dt} + ae^{at}y = be^{at} \qquad (*)$$

- Why is this a good idea? Using the chain rule, we know that

- Therefore, integrating both sides of $(*)$ with respect to $t$, we obtain

- Rearranging terms like in the homogeneous case, we get

- The **complementary function** is
  - Note this is the solution to the homogeneous version of the differential equation
- The **particular integral** is
- The **general solution** is
- We can find $A$ by setting $t = 0$ in the general solution:

- The **definite solution** is

**Example 2.** Solve the equation $\frac{dy}{dt} + 4y = 0$, with the initial condition $y(0) = 1$. Verify your solution.

- Note that for Example 2, we could have used the formula we derived for either the homogenous or the nonhomogeneous case

**Example 3.** Solve the equation $\frac{dy}{dt} + 2y = 6$, with the initial condition $y(0) = 10$. Verify your solution.

## 5 A market equilibrium model with price dynamics

- Market with single commodity
- Variables:

  $Q_d$ = quantity demanded $\quad$ $Q_s$ = quantity supplied $\quad$ $P$ = unit price

- Equations:

$$
\begin{aligned}
Q_d &= \alpha - \beta P && (\alpha, \beta > 0) \\
Q_s &= -\gamma + \delta P && (\gamma, \delta > 0) \\
\frac{dP}{dt} &= j(Q_d - Q_s) && (j > 0) \qquad (1)
\end{aligned}
$$

- $j$ is a constant **adjustment coefficient**
- What does (1) say about how the price changes over time?
  - $Q_d > Q_s \Rightarrow$
  - $Q_d < Q_s \Rightarrow$
  - $Q_d = Q_s \Rightarrow$

**Example 4.** Combine the three equations above to form a first-order linear nonhomogeneous differential equation. Solve for $P(t)$.