SM286A – Mathematics for Economics Fall 2015
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Lesson 13. Exact Differential Equations

1 The total differential

- The **total differential** of $F(y, t)$ is

Example 1. Find the total differential of $F(y, t) = y^2 t$.

2 Exact differential equations

- If we set the total differential of a function $F(y, t)$ to zero, we get an **exact differential equation**:

- For example, using $F(y, t) = y^2 t$ from Example 1,

  is an example of an exact differential equation

3 Detecting exact differential equations

- A differential equation
$$M\,dy + N\,dt = 0 \qquad (*)$$
  is exact if and only if $M = \partial F/\partial y$ and $N = \partial F/\partial t$ for some function $F(y, t)$

- A simple test: $(*)$ is exact if and only if $\partial M/\partial t = \partial N/\partial y$

Example 2. Is the differential equation $2t\,dy + y\,dt = 0$ exact?

4 Solving exact differential equations

- An exact differential equation is of the form $dF(y, t) = 0$ for some function $F(y, t)$

$\Rightarrow$ Its general solution must be of the form

- We can then use this equation to solve for $y(t)$, if desired

Method for solving an exact differential equation

$$M\,dy + N\,dt = 0$$

Step 0. Check that it is exact!

Step 1. Use the fact that $M = \partial F/\partial y$ to find a preliminary version of F:

$$F(y, t) = \int M\,dy + \psi(t)$$

Step 2. Find $\partial F/\partial t$ based on the preliminary version of F found in Step 1.
Use this and the fact that $N = \partial F/\partial t$ to find $d\psi/dt$.

Step 3. Integrate $d\psi/dt$ found in Step 2 to find $\psi(t)$.

Step 4. Combine Steps 1 and 3 to find $F(y, t)$. Set $F(y, t) = c$. Solve for $y(t)$ if desired.

Example 3. Solve the exact differential equation $(t + 2y)\,dy + (y + 3t^2)\,dt = 0$.

Example 4. Solve the exact differential equation $2yt\,dy + y^2\,dt = 0$.

5 Integrating factors

- Sometimes we can convert an inexact differential equation into an exact one by multiplying both sides of the equation by an **integrating factor**

Example 5. In Example 2, we showed that the differential equation $2t\,dy + y\,dt = 0$ is inexact. Show that y is an integrating factor for this equation.

- An integrating factor may not always exist
- Integrating factors may be hard to find

6 Solving first-order linear differential equations with variable coefficients

- Last time, we stated that the differential equation

$$\frac{dy}{dt} + u(t)y = w(t) \quad \text{has general solution} \quad y(t) = e^{-\int u\,dt}\left(A + \int we^{\int u\,dt}\,dt\right)$$

- Let's derive this general solution
- We can rewrite the differential equation as

$$dy + (uy - w)\,dt = 0$$

- It turns out that $e^{\int u\,dt}$ is an integrating factor, so we have the exact differential equation

$$e^{\int u\,dt}dy + e^{\int u\,dt}(uy - w)\,dt = 0$$

- Let's check that this equation is in fact exact:

- Now we can apply the four-step method: