SM286A – Mathematics for Economics Fall 2015
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Lesson 14. Nonlinear Differential Equations of the First Order and First Degree

0 Warm up

Example 1. Consider the differential equation $\frac{dy}{dt} = \frac{t}{3y^2}$.

a. Rewrite this equation in the form $M\,dy + N\,dt = 0$.
b. Show that the differential equation you obtained in part a is exact.
c. Solve this differential equation using the solution method from Lesson 13.

1 Overview

- Today, we will consider differential equations that are
  - **first-order**: only the first derivative dy/dt of y appears
  - **first-degree**: dy/dt is only raised to the first power
  - **nonlinear**: no restrictions on the power of y and t
- Some of these differential equations can be solved with relative ease, as we will see today
  - e.g. exact differential equations

2 Separable variables

$$M\,dy + N\,dt = 0 \qquad (*)$$

- The variables in the differential equation $(*)$ are **separable** if
  - M is a function of y alone
  - N is a function of t alone
- For example, the differential equation in Example 1 had separable variables: $3y^2\,dy - t\,dt = 0$
- Differential equations with separable variables can be solved using basic integration techniques

Example 2. Solve the equation $3y^2\,dy - t\,dt = 0$ using basic integration techniques.

- Sometimes an equation doesn't have separable variables, but it can transformed into one that does
- Consider the equation $e^{-2t}\,dy - \frac{1}{y}\,dt = 0$
- To make this equation have separable variables:

Example 3. Solve the equation $y\,dy - e^{2t}\,dt = 0$ using basic integration techniques.

3 Bernoulli equations

- A **Bernoulli equation** is a differential equation of the form

$$\frac{dy}{dt} + Ry = Ty^m \qquad (m \neq 0, 1) \qquad \text{where } R \text{ and } T \text{ are functions of } t$$

- We can transform a Bernoulli equation into an equivalent linear differential equation, which we know how to solve

- Let $z = y^{1-m}$. Then $\frac{dz}{dt} =$

- Mutiplying both sides of the Bernoulli equation by $(1-m)y^{-m}$, we obtain

- Therefore, the Bernoulli equation is equivalent to

- Note that this is an equation of the form

$$\frac{dy}{dt} + u(t)y = w(t) \quad \text{which has general solution} \quad y(t) = e^{-\int u\,dt}\left(A + \int we^{\int u\,dt}\,dt\right)$$

Example 4. Solve the Bernoulli equation $\frac{dy}{dt} + ty = 3ty^2$.

Example 5. Solve the Bernoulli equation $\frac{dy}{dt} + y = y^3$.