SM286A – Mathematics for Economics Fall 2015
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Lesson 15. Qualitative Analysis of Differential Equations

1 Overview

- So far, we have solved differential equations **quantitatively**
  - e.g. Given a differential equation $M\,dy + N\,dt = 0$, find a formula for $y(t)$
- Sometimes, we may not be able to find a formula for $y(t)$
- In these cases, we may still be able to determine the **qualitative** behavior of $y(t)$
  - e.g. When is $y(t)$ increasing or decreasing? Does $y(t)$ converge?

2 The phase diagram

- An **autonomous** differential equation is an equation of the form

$$\frac{dy}{dt} = f(y) \qquad \text{where } f \text{ is a function of } y \text{ } \underline{\text{alone}} \qquad (*)$$

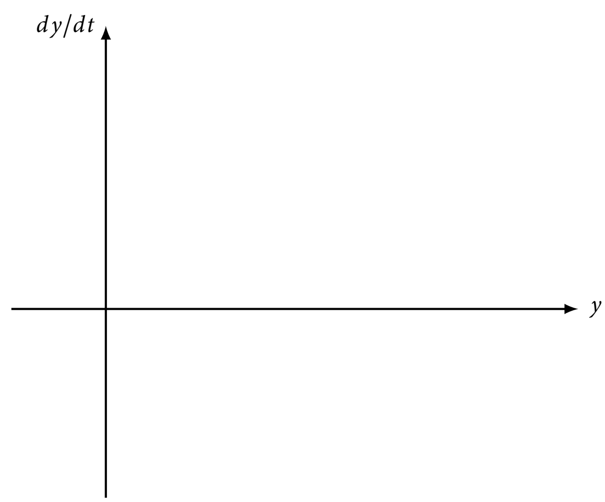
- For example, from the dynamic market equilibrium model in Lesson 11:

$$\frac{dP}{dt} = j(\alpha + \gamma) - j(\beta + \delta)P \qquad \text{is autonomous}$$

- The **phase line** is the graph of dy/dt vs. y (i.e., the graph of f)

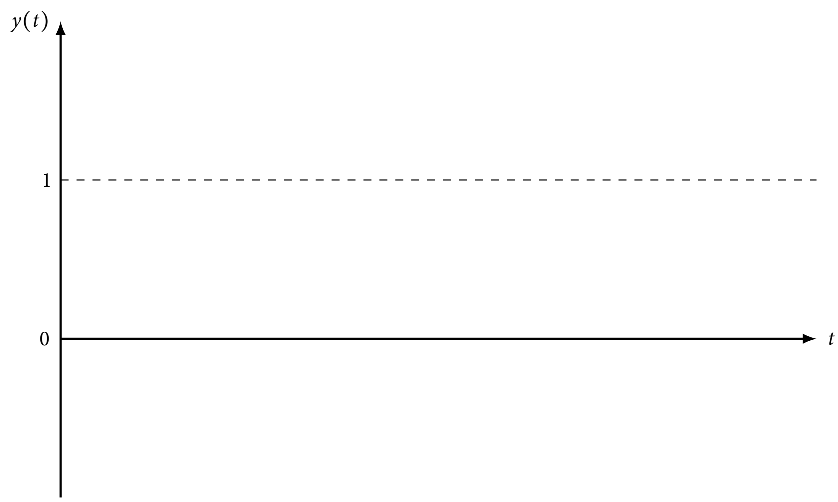
Example 1.

a. Plot the phase line for $dy/dt = y - y^2$.
b. When $y = 1/2$, is $y(t)$ increasing or decreasing in t?
c. When $y = 2$, is $y(t)$ increasing or decreasing in t?
d. When is $dy/dt = 0$?



- To complete the **phase diagram**:
  - Draw arrows on the horizontal axis indicating the direction in which y is moving
    - When $dy/dt > 0$, $y(t)$ is
    - When $dy/dt < 0$, $y(t)$ is
  - Mark the **equilibrium points**, or where y is stationary over time ($dy/dt = 0$)

Example 2. Using the phase diagram, draw approximately how y behaves over time, starting at $y = -0.1$, $y = 0.1$, $y = 0.5$, $y = 1.5$.



- If the slope at an equilibrium point is:
  - positive, then the equilibrium is
  - negative, then the equilibrium is

Example 3. Plot the phase line for $dy/dt = 3 - 2y$. What are the equilibrium points? Are the equilibrium points dynamically stable or unstable?