SM286A – Mathematics for Economics Fall 2015
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Lesson 16. The Solow Growth Model

0 Warm up

Example 1.

Consider a function $f(x)$ with $df/dx > 0$ and $df^2/dx^2 < 0$. Draw what the graph of $y = f(x)$ might look like.

1 Overview

- The **Solow growth model** is an economic model of long-run economic growth
- This model attempts to explain long-run economic growth by looking at
  - capital accumulation
  - labor growth

2 The model

- Notation:
  - $\dot{y} = dy/dt$ for any function $y(t)$
  - $f_u = \partial f/\partial u$, $f_{uu} = \partial^2 f/\partial u^2$ for any function $f(u,v)$
- Variables: K = capital, L = labor Q = output
- Parameters:

$$s = \text{marginal propensity to save} \quad (0 < s < 1)$$
$$\lambda = \text{rate of growth of labor} \quad (\lambda > 0)$$

- Model:

$$Q = f(K, L) \quad (K, L > 0)$$

$$\dot{K} = sQ$$

$$\dot{L} = \lambda L$$

- Assumptions:
  - f is homogeneous: $f(aK, aL) = af(K, L)$
  - $f_K > 0$, $f_L > 0$: output is increasing in capital and labor
  - $f_{KK} < 0$, $f_{LL} < 0$: diminishing marginal output in capital and labor

3 Simplifying the model

- Define

$$\phi(k) = \frac{Q}{L} \qquad \text{where } k = \frac{K}{L}$$

- It's not too hard to show that

$$f_K > 0 \quad \Rightarrow \quad \frac{d\phi}{dk} > 0 \qquad\qquad \text{and} \qquad\qquad f_{KK} < 0 \quad \Rightarrow \quad \frac{d^2\phi}{dk^2} < 0$$

- By combining the definition of ϕ and the model equations, we obtain the differential equation

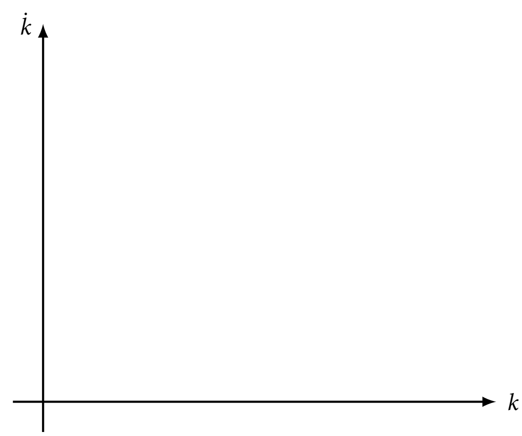
$$\dot{k} = s\phi(k) - \lambda k \tag{S}$$

- This is where we will start our analysis

4 Qualitative analysis

- To draw the phase line of (S), let's first draw the two individual terms on the RHS:

- Now let's use this information to draw the phase line:



- So there is an equilibrium $k = \bar{k}$ that is
- What does this mean economically?
  - At $k = \bar{k}$, we have $\dot{k} = 0$, or in words:
  - Therefore, in the long run, capital must grow at the same rate as labor
  - Recall that we defined $\phi(k) = Q/L$. So, at $k = \bar{k}$, we have also that

$$Q = L\phi(\bar{k})$$

  - Therefore, in the long run, output must also grow at the same rate as labor

5 Quantitative analysis

Example 2. Suppose the production function in the Solow growth model is $f(K, L) = K^{3/4}L^{1/4}$.

a. Show that f is homogeneous: $f(aK, aL) = af(K, L)$.
b. Find $\phi(k)$.
c. Write the differential equation (S). Solve the equation for k. *Hint.* It is a Bernoulli equation!
d. What does k converge to as $t \to \infty$?