Lesson 17. Discrete Time and Difference Equations

0 Warm up

- Let's find a simple formula for the sum $S = 1 + \alpha + \alpha^2 + \cdots + \alpha^{t-1}$, assuming $\alpha \neq 1$.
- First, $\alpha S =$ [ ]
- Therefore, $S - \alpha S =$ [ ]. Note that $S - \alpha S = (1 - \alpha)S$.
- Divide both sides of the above equation by $(1 - \alpha)$ to obtain another expression for S:

[ ]

1 Overview

- Differential equations deal with dynamics in **continuous time**
  - e.g. $P(t)$ is the price of a product at any real-valued time t: $P(0)$, $P(1.083)$, $P(623.2)$, etc.
  - These dynamics are embodied in derivatives
- Sometimes it is more convenient to model dynamics in **discrete time**
  - e.g. price of a product at distinct time **periods**: every month, every year, etc.
  - These dynamics are described by **differences**

2 Difference equations

- We want to study how a sequence of values $y_0, y_1, y_2, \ldots$ changes over time
- A **difference equation** expresses a value of a sequence as a function of other terms in the sequence
- Examples:

$$y_{t+1} = y_t + 2 \qquad\qquad y_{t+1} = 0.9y_t$$

- The **first difference** of y at period t is $\Delta y_t = y_{t+1} - y_t$
- We can rewrite the difference equations above as:

[ ]

- We will concentrate on **first-order** difference equations: equations with only first differences, or one-period time lags (i.e. y_t and y_{t+1})
- The **solution** of a difference equation is a formula defining the values of y_t in every time period t

3 Solving first-order difference equations by iteration

- First, a naïve approach
- Start with y_1, then find $y_2, y_3, \ldots$ using the difference equation
- Infer a pattern to get a formula for y_t for any period t

Example 1. Solve the difference equation $y_{t+1} = y_t + 2$, assuming an initial value $y_0 = 15$. Verify your answer.

Example 2. Solve the difference equation $y_{t+1} = 0.9y_t$, assuming an initial value $y_0 = 4$. Verify your answer.

4 A general method for solving first-order difference equations

$$\text{Given} \quad y_{t+1} + ay_t = c, \qquad \text{what is} \quad y_t \quad ?$$

- By iteration, we obtain

$$
\begin{aligned}
y_1 &= -ay_0 + c \\
y_2 &= -ay_1 + c = -a(-ay_0 + c) + c = (-a)^2 y_0 + (1-a)c \\
y_3 &= -ay_2 + c = -a((-a)^2 y_0 + (1-a)c) + c = (-a)^3 y_0 + (1 + (-a) + (-a)^2)c \\
&\vdots \\
y_t &= (-a)^t y_0 + (1 + (-a) + (-a)^t + \cdots + (-a)^{t-1})c
\end{aligned}
$$

- If $a = -1$, then the solution is

- If $a \neq -1$, then the solution is

or equivalently

Example 3. Solve the difference equation $y_{t+1} - 5y_t = 1$, assuming $y_0 = 7/4$. Verify your answer.