SM286A – Mathematics for Economics Fall 2015
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Lesson 20. Review: Optimization of Functions with 1 or 2 Variables, Determinants

1 Optimization of a function of a single variable

- Let's start with some review of Calculus I
- Let f be a function of a single variable
- $f(a)$ is a **local minimum** of f if $f(a) \leq f(x)$ for all x "near" a
- $f(a)$ is a **local maximum** of f if $f(a) \geq f(x)$ for all x "near" a
- a is a **critical point** of f if either

  (i) $f'(a) = 0$ or (ii) $f'(a)$ does not exist

- Local optima must occur at critical points
- Finding local optima:
  - Let's assume f' and f'' always exist
  - Let a be a critical point of f – in this case, that means $f'(a) = 0$
  - Then:

$$\begin{array}{lll} \text{if} & f''(a) > 0, & \text{then } f(a) \text{ is a local minimum of } f \\ \text{if} & f''(a) < 0, & \text{then } f(a) \text{ is a local maximum of } f \end{array}$$

Example 1. Let $f(x) = x^3 - x = x(x-1)(x+1)$. Sketch the graph of $f(x)$. Find the local optima of f.

Example 2. Let $f(x) = 12 + 4x - x^2$. Sketch the graph of $f(x)$. Find the local optima of f.

2 Optimization of a function of two variables

- Next, let's review some Calculus III
- Let f be a function of two variables
- $f(a,b)$ is a **local minimum** of f if $f(a,b) \leq f(x,y)$ for all (x,y) "near" (a,b)
- $f(a,b)$ is a **local maximum** of f if $f(a,b) \geq f(x,y)$ for all (x,y) "near" (a,b)
- (a,b) is a **critical point** of f if either

  (i) $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or (ii) $f_x(a,b)$ or $f_y(a,b)$ does not exist

- Local optima must occur at critical points

- Finding local optima:
  - Let's assume f_x, f_y, f_{xx}, f_{yy}, and f_{xy} always exist
  - Let (a,b) be a critical point of f – in this case, that means $f_x(a,b) = 0$ and $f_y(a,b) = 0$
  - Define $D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$
  - Then:

$$\begin{array}{llll} \text{if} & D > 0 & \text{and} \quad f_{xx}(a,b) > 0, & \text{then } f(a,b) \text{ is a local minimum of } f \\ \text{if} & D > 0 & \text{and} \quad f_{xx}(a,b) < 0, & \text{then } f(a,b) \text{ is a local maximum of } f \\ \text{if} & D < 0, & \text{then } f(a,b) \text{ is a } \textbf{saddle point} \text{ of } f & \end{array}$$

Example 3. Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Find the local optima of f.

Example 4. Let $f(x,y) = x^3 - 12xy + 8y^3$. Find the local optima of f.

3 Determinants

- Finally, let's review Lesson 7
- The **determinant** $|A|$ of square matrix A is a uniquely defined scalar associated with A
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = ad - bc$
- For larger matrices, use **Laplace expansion**

Example 5. Find the following determinants:

$$\begin{vmatrix} 2 & -1 \\ 3 & 8 \end{vmatrix} \qquad \begin{vmatrix} 0 & -2 & 5 \\ 4 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & -2 & 5 \\ 1 & 1 & 0 \\ 4 & 1 & 3 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 4 & 3 \\ 0 & -1 & 2 & 5 \\ -2 & 1 & 0 & 3 \end{vmatrix}$$