Example 2. Find the local optima of $f(x_1, x_2, x_3) = 2x_1^2 + 4x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + 2$. Is f strictly convex or concave? If so, what can you conclude about the local optima that you found?

- First, we use the first-order necessary condition to find critical points. The gradient of f is

  $$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 + x_2 + x_3 \\ x_1 + 8x_2 \\ x_1 + 2x_3 \end{bmatrix}$$

  Therefore any critical points must satisfy:

  $$\nabla f(x_1, x_2, x_3) = 0 \quad \Leftrightarrow \quad \begin{aligned} 4x_1 + x_2 + x_3 &= 0 \\ x_1 + 8x_2 &= 0 \\ x_1 + 2x_3 &= 0 \end{aligned}$$

  Solving this system of equations, we find 1 critical point: $(x_1, x_2, x_3) = (0, 0, 0)$.

- Next, we use the second-order sufficient condition to classify the critical points we found. The Hessian matrix of f is

  $$H(x_1, x_2, x_3) = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

  Therefore, the Hessian matrix at $(x_1, x_2, x_3) = (0, 0, 0)$ is

  $$H(0, 0, 0) = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

  and the leading principal minors of $H(0, 0, 0)$ are

  $$|H_1| = 4 > 0 \qquad |H_2| = 31 > 0 \qquad |H_3| = 54 > 0$$

  So, $f(0, 0, 0) = 2$ is a local minimum.

- Finally, we determine whether f is strictly convex or concave. Note that <u>for all values of (x_1, x_2, x_3)</u>, the leading principal minors of $H(x_1, x_2, x_3)$ are

  $$|H_1| = 4 > 0 \qquad |H_2| = 31 > 0 \qquad |H_3| = 54 > 0$$

  Therefore, f is strictly convex, and $f(0, 0, 0) = 2$ is in fact an absolute minimum.

Example 3. Find the local optima of $f(x_1, x_2, x_3) = -x_1^2 - (x_1 + x_2)^2 - (x_1 + x_3)^2$. Is f strictly convex or concave? If so, what can you conclude about the local optima that you found?

- First, we use the first-order necessary condition to find critical points. The gradient of f is

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} -6x_1 - 2x_2 - 2x_3 \\ -2x_1 - 2x_2 \\ -2x_1 - 2x_3 \end{bmatrix}$$

  Therefore any critical points must satisfy:

$$\nabla f(x_1, x_2, x_3) = 0 \qquad \Leftrightarrow \qquad \begin{aligned} -6x_1 - 2x_2 - 2x_3 &= 0 \\ -2x_1 - 2x_2 &= 0 \\ -2x_1 - 2x_3 &= 0 \end{aligned}$$

  Solving this system of equations, we find 1 critical point: $(x_1, x_2, x_3) = (0, 0, 0)$.

- Next, we use the second-order sufficient condition to classify the critical points we found. The Hessian matrix of f is

$$H(x_1, x_2, x_3) = \begin{bmatrix} -6 & -2 & -2 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

  Therefore, the Hessian matrix at $(x_1, x_2, x_3) = (0, 0, 0)$ is

$$H(0, 0, 0) = \begin{bmatrix} -6 & -2 & -2 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

  and the leading principal minors of $H(0, 0, 0)$ are

$$|H_1| = -6 < 0 \qquad |H_2| = 8 > 0 \qquad |H_3| = -8 < 0$$

  So, $f(0, 0, 0) = 0$ is a local minimum.

- Finally, we determine whether f is strictly convex or concave. Note that <u>for all values of (x_1, x_2, x_3)</u>, the leading principal minors of $H(x_1, x_2, x_3)$ are

$$|H_1| = -6 < 0 \qquad |H_2| = 8 > 0 \qquad |H_3| = -8 < 0$$

  Therefore, f is strictly convex, and $f(0, 0, 0) = 0$ is in fact an absolute minimum.

Example 4. Find the local optima of $f(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^4 - 4x_1x_3 - 2x_2$. Is f strictly convex or concave? If so, what can you conclude about the local optima that you found? *Hint.* What happens when $x_3 = 0$?

- First, we use the first-order necessary condition to find critical points. The gradient of f is

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 - 4x_3 \\ 2x_2 - 2 \\ 4x_3^3 - 4x_1 \end{bmatrix}$$

  Therefore any critical points must satisfy:

$$\nabla f(x_1, x_2, x_3) = 0 \quad \Leftrightarrow \quad \begin{aligned} 4x_1 - 4x_3 &= 0 \\ 2x_2 - 2 &= 0 \\ 4x_3^3 - 4x_1 &= 0 \end{aligned}$$

  Solving this system of equations, we find 3 critical points:

$$(x_1, x_2, x_3) = (0, 1, 0) \qquad (x_1, x_2, x_3) = (-1, 1, -1) \qquad (x_1, x_2, x_3) = (1, 1, 1).$$

- Next, we use the second-order sufficient condition to classify the critical points we found. The Hessian matrix of f is

$$H(x_1, x_2, x_3) = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 0 & 12x_3^2 \end{bmatrix}$$

  1. $\underline{(x_1, x_2, x_3) = (0, 1, 0)}$. The Hessian matrix at $(x_1, x_2, x_3) = (0, 1, 0)$ is

$$H(0, 1, 0) = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

     and the leading principal minors of $H(0, 1, 0)$ are

$$|H_1| = 4 > 0 \qquad |H_2| = 8 > 0 \qquad |H_3| = -8 < 0.$$

     Therefore, $(x_1, x_2, x_3) = (0, 1, 0)$ is a saddle point.
  2. $\underline{(x_1, x_2, x_3) = (-1, 1, -1)}$. The Hessian matrix at $(x_1, x_2, x_3) = (-1, 1, -1)$ is

$$H(-1, 1, -1) = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 0 & 12 \end{bmatrix}$$

     and the leading principal minors of $H(-1, 1, -1)$ are

$$|H_1| = 4 > 0 \qquad |H_2| = 8 > 0 \qquad |H_3| = 88 > 0.$$

     Therefore, $(x_1, x_2, x_3) = (-1, 1, -1)$ is a local minimum.
  3. $\underline{(x_1, x_2, x_3) = (1, 1, 1)}$. The Hessian matrix at $(x_1, x_2, x_3) = (1, 1, 1)$ is

$$H(1, 1, 1) = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 0 \\ -4 & 0 & 12 \end{bmatrix}$$

and the leading principal minors of $H(1,1,1)$ are

$$|H_1| = 4 > 0 \qquad |H_2| = 8 > 0 \qquad |H_3| = 88 > 0.$$

Therefore, $(x_1, x_2, x_3) = (1,1,1)$ is a local minimum.

- Finally, we determine whether f is strictly convex or concave. From the previous step, we see that we do not have $|H_1| > 0$, $|H_2| > 0$ and $|H_3| > 0$ for all values of (x_1, x_2, x_3): for example, the Hessian matrix at $(x_1, x_2, x_3) = (0,1,0)$ does not satisfy this condition. Therefore, f is not strictly convex.

  We also see from the previous step that we do not have $|H_1| < 0$, $|H_2| > 0$ and $|H_3| < 0$ for all values of (x_1, x_2, x_3): for example, the Hessian matrix at $(x_1, x_2, x_3) = (0,1,0)$ does not satisfy this condition. Therefore, f is not strictly concave.