SM286A – Mathematics for Economics Fall 2015
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Lesson 22. Absolute Optima, Convexity and Concavity

1 Saddle points

- From the last lesson...
- Let f be a function of n variables
- Assume all of the first and second partial derivatives exist
- **First-order necessary condition.** $\nabla f(a_1, \dots, a_n) = 0$
- **Second-order sufficient condition.**

  (i) local minimum: $|H_1| > 0,\ |H_2| > 0,\ \dots,\ |H_n| > 0$
  (ii) local maximum: $|H_1| < 0,\ |H_2| > 0,\ |H_3| < 0,$ etc.

- What if (i) and (ii) don't hold?
  - If $H(a_1, \dots, a_n)$ is nonsingular, then $(a_1, \dots, a_n)$ is a **saddle point** (and therefore not a local optimum)
  - If $H(a_1, \dots, a_n)$ is singular, then further investigation is required
- Recall that $H(a_1, \dots, a_n)$ is singular if $|H(a_1, \dots, a_n)| =$ ☐

2 Absolute optima

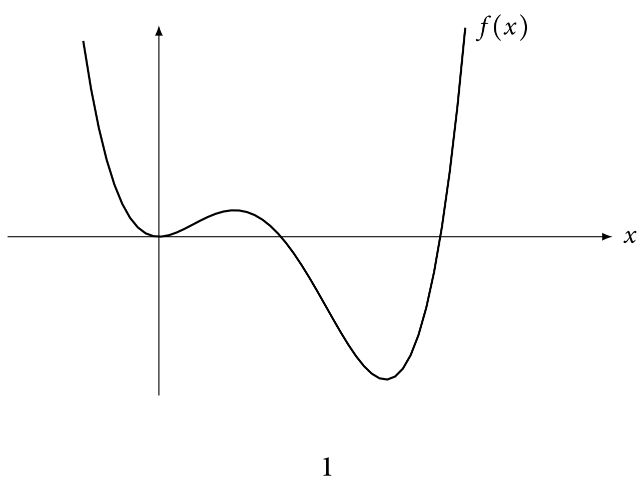
- $f(a_1, a_2, \dots, a_n)$ is an **absolute minimum** of f if

$$f(a_1, a_2, \dots, a_n) \leq f(x_1, x_2, \dots, x_n) \quad \text{for all } (x_1, x_2, \dots, x_n)$$

- $f(a_1, a_2, \dots, a_n)$ is an **absolute maximum** of f if

$$f(a_1, a_2, \dots, a_n) \geq f(x_1, x_2, \dots, x_n) \quad \text{for all } (x_1, x_2, \dots, x_n)$$

- An absolute optimum is also a local optimum
- A local optimum is not necessarily an absolute optimum!



- Unfortunately, absolute optima are typically very hard to find
- In certain cases, a local optimum is always an absolute optimum...

3 Strictly convex and concave functions

- f is **strictly convex** if and only if

$$|H_1| > 0 \quad |H_2| > 0 \quad |H_3| > 0 \quad \cdots \qquad \text{for all } (x_1, \dots, x_n)$$

  - If f is strictly convex, then every local minimum is also an absolute minimum
  - If f is strictly convex and has an absolute minimum, the point that achieves that minimum is unique

- f is **strictly concave** if and only if

$$|H_1| < 0 \quad |H_2| > 0 \quad |H_3| < 0 \quad \cdots \qquad \text{for all } (x_1, \dots, x_n)$$

  - If f is strictly concave, then every local maximum is also an absolute maximum
  - If f is strictly concave and has an absolute maximum, the point that achieves that maximum is unique

Example 1. Show that $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ is strictly convex.

- There are other weaker notions of convexity and concavity that also imply that local optima are absolute optima
  - See your textbook for details

4 Practice!

Example 2. Find the local optima of $f(x_1, x_2, x_3) = 2x_1^2 + 4x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + 2$. Is f strictly convex or concave? If so, what can you conclude about the local optima that you found?

Example 3. Find the local optima of $f(x_1, x_2, x_3) = -x_1^2 - (x_1 + x_2)^2 - (x_1 + x_3)^2$. Is f strictly convex or concave? If so, what can you conclude about the local optima that you found?

Example 4. Find the local optima of $f(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^4 - 4x_1x_3 - 2x_2$. Is f strictly convex or concave? If so, what can you conclude about the local optima that you found? *Hint.* What happens when $x_3 = 0$?