Lesson 25. Optimization with Equality Constraints, cont.

1 Overview

- Last time: the Lagrange multiplier method for optimization problems with one equality constraint
- Today: multiple equality constraints

2 The Lagrange multiplier method – m equality constraints

$$\begin{aligned} \text{minimize/maximize} \quad & f(x_1, \dots, x_n) \\ \text{subject to} \quad & g_1(x_1, \dots, x_n) = c_1 \\ & \vdots \\ & g_m(x_1, \dots, x_n) = c_m \end{aligned}$$

- **Step 1.** Introduce Lagrange multipliers $\lambda_1, \dots, \lambda_m$ for each equality constraint and form the Lagrangian function Z:

$$Z(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = f(x_1, \dots, x_n) + \lambda_1\big[c_1 - g_1(x_1, \dots, x_n)\big] + \cdots + \lambda_m\big[c_m - g_m(x_1, \dots, x_n)\big]$$

- **Step 2.** Find the critical points by solving the system of equations implied by the first-order necessary condition:

$$\nabla Z(x_1, \dots, x_n, \lambda_1, \dots, \lambda_m) = 0$$

- **Step 3.** Classify each critical point as a local minimum or local maximum by applying the second-order sufficient condition:
  - The bordered Hessian matrix $\overline{H}$ is

$$\overline{H} = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ 0 & 0 & \dots & 0 & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \dots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \dots & \frac{\partial g_m}{\partial x_n} \\ \hline \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \dots & \frac{\partial g_m}{\partial x_1} & \frac{\partial^2 Z}{\partial x_1^2} & \frac{\partial^2 Z}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 Z}{\partial x_1 x_n} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \dots & \frac{\partial g_m}{\partial x_2} & \frac{\partial^2 Z}{\partial x_2 \partial x_1} & \frac{\partial^2 Z}{\partial x_2^2} & \dots & \frac{\partial^2 Z}{\partial x_2 x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \dots & \frac{\partial g_m}{\partial x_n} & \frac{\partial^2 Z}{\partial x_n \partial x_1} & \frac{\partial^2 Z}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 Z}{\partial x_n^2} \end{array}\right]$$

  - The ith bordered leading principal minor of $\overline{H}$ — denoted by $|\overline{H}_i|$ — is the determinant of the square submatrix formed by the first $m + i$ rows and columns of $\overline{H}$

- Let $(a_1, \dots, a_n)$ be a critical point found in Step 2. Then
  (i) $f(a_1, a_2, \dots, a_n)$ is a local minimum if

$$|\overline{H}_{m+1}|, |\overline{H}_{m+2}|, \dots, |\overline{H}_n| \quad \text{all have the same sign as } (-1)^m$$

  (ii) $f(a_1, a_2, \dots, a_n)$ is a local maximum if

$$|\overline{H}_{m+1}| \text{ has the same sign as } (-1)^{m+1} \quad \text{and} \quad |\overline{H}_{m+1}|, |\overline{H}_{m+2}|, \dots, |\overline{H}_n| \text{ alternate in sign}$$

Example 1. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 12 \\ & x_1^2 + x_2^2 - x_3 = 0 \end{aligned}$$

- In this problem, $n =$ [ ] and $m =$ [ ]

Step 1. Introduce the Lagrange multipliers $\lambda_1, \dots, \lambda_m$ and form the Lagrangian function Z.

- The Lagrangian function Z is

Step 2. Find the critical points.

- The gradient of Z is

- The first-order necessary condition tells us that the critical points must satisfy

- Solving this system of equations, we find that there are two critical points:

Step 3. Classify the critical points as a local minimum or local maximum.

- The bordered Hessian is

- The bordered Hessian at the critical point $(x_1, x_2, x_3, \lambda_1, \lambda_2) = (2, 2, 8, 4/5, -1/5)$ is

- The bordered leading principal minors $|\overline{H}_{m+1}|, |\overline{H}_{m+2}|, \dots$ of $\overline{H}(2, 2, 8, 4/5, -1/5)$ are

- Therefore,

- The bordered Hessian at the critical point $(x_1, x_2, x_3, \lambda_1, \lambda_2) = (-3, -3, 18, 6/5, 1/5)$ is

- The bordered leading principal minors $|\overline{H}_{m+1}|, |\overline{H}_{m+2}|, \ldots$ of $\overline{H}(-3, -3, 18, 6/5, 1/5)$ are

- Therefore,

Example 2. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} \quad & 3x_1 + x_2 + x_3 = 5 \\ & x_1 + x_2 + x_3 = 1 \end{aligned}$$