SM286A – Mathematics for Economics Fall 2015
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Lesson 26. Applications of Optimization with Equality Constraints

Example 1. Eli Orchid manufactures its newest pharamceutical product, Med-X, using its two patented processes. Process 1 costs \$1,000 per batch, and Process 2 costs \$250 per batch. These two processes work in tandem: if Eli Orchid uses x_1 batches of Process 1 and x_2 batches of Process 2, it produces $10\sqrt{x_1 x_2}$ liters of Med-X. Eli Orchid wants to find the least costly way of producing 100 liters of Med-X.

a. Using the variables x_1 and x_2 defined above, write cost as a function of x_1 and x_2: $c(x_1, x_2) = \dots$
b. Using the variables x_1 and x_2 defined above, write an equality constraint that represents that Eli Orchid must produce 100 liters of Med-X.
c. Find the local optima of the cost function c you wrote in part a, subject to the equality constraint you wrote in part b.
d. How many batches of Process 1 and Process 2 should Eli Orchid use? What is the corresponding cost?

a. $c(x_1, x_2) = 1000x_1 + 250x_2$

b. $10\sqrt{x_1 x_2} = 100$

c. $Z(x_1, x_2, \lambda) = 1000x_1 + 250x_2 + \lambda\left[100 - 10x_1^{1/2}x_2^{1/2}\right]$

$$\nabla Z(x_1, x_2, \lambda) = \begin{bmatrix} 1000 - 5x_1^{-1/2}x_2^{1/2}\lambda \\ 250 - 5x_1^{1/2}x_2^{-1/2}\lambda \\ 100 - 10x_1^{1/2}x_2^{1/2} \end{bmatrix}$$

Solve $\nabla Z = 0$

$\Rightarrow$ 1 critical pt: $(x_1, x_2, \lambda) = (5, 20, 100)$

$$\bar{H}(x_1, x_2, \lambda) = \begin{bmatrix} 0 & 5x_1^{-1/2}x_2^{1/2} & 5x_1^{1/2}x_2^{-1/2} \\ 5x_1^{-1/2}x_2^{1/2} & \frac{5}{2}x_1^{-3/2}x_2^{1/2}\lambda & -\frac{5}{2}x_1^{-1/2}x_2^{-1/2}\lambda \\ 5x_1^{1/2}x_2^{-1/2} & -\frac{5}{2}x_1^{-1/2}x_2^{-1/2}\lambda & \frac{5}{2}x_1^{1/2}x_2^{-3/2}\lambda \end{bmatrix}$$

$$\Rightarrow \bar{H}(5, 20, 100) = \begin{bmatrix} 0 & 10 & 5/2 \\ 10 & 100 & -25 \\ 5/2 & -25 & \frac{25}{4} \end{bmatrix}$$

So, $|\bar{H}_2| = -2500$

$\Rightarrow c(5, 20) = 10{,}000$ is a local minimum

d. (Assuming a local optimum is good enough) Eli Orchid should use 50 and 200 batches of Processes 1 and 2, respectively, at a cost of 10,000

Example 2. Suppose that you are interested in dividing your savings between three mutual funds with expected returns of 10%, 10% and 15%, respectively. You want to minimize risk while achieving an expected return of 12%. To measure risk, use the *variance* of the return on investment: when a fraction x of your savings is invested in Fund 1, y in Fund 2, and z in Fund 3, the variance of the return has been calculated to be

$$v(x, y, z) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$$

a. Consider the equality constraints below. Why do these constraints make sense for this problem?

expected return of portfolio must be 12% $\longrightarrow$ $1.10x + 1.10y + 1.15z = 1.12$ (1)

$x + y + z = 1$ $\longleftarrow$ fractions of savings must sum up to 1. (2)

b. Find the local optima of the variance v, subject to the equality constraints given in part a.

c. How much should you invest in the three mutual funds?

b. $Z(x,y,z,\lambda,\mu) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$
$+ \lambda[1.12 - 1.10x - 1.10y - 1.15z] + \mu[1 - x - y - z]$

$$\nabla Z(x,y,z,\lambda,\mu) = \begin{bmatrix} 800x + 200y - 1.1\lambda - \mu \\ 1600y + 200x + 400z - 1.1\lambda - \mu \\ 3200z + 400y - 1.15\lambda - \mu \\ 1.12 - 1.10x - 1.10y - 1.15z \\ 1 - x - y - z \end{bmatrix}$$

Solve $\nabla Z = 0$

$\Rightarrow$ 1 critical point: $(x, y, z, \lambda, \mu) = (0.5, 0.1, 0.4, 18000, -19380)$

$$\bar{H}(x,y,z,\lambda,\mu) = \begin{bmatrix} 0 & 0 & 1.1 & 1.1 & 1.15 \\ 0 & 0 & 1 & 1 & 1 \\ 1.1 & 1 & 800 & 200 & 0 \\ 1.1 & 1 & 200 & 1600 & 400 \\ 1.15 & 1 & 0 & 400 & 3200 \end{bmatrix} = \bar{H}(0.5, 0.1, 0.4, 18000, -19380)$$

$\Rightarrow |\bar{H}_3| = 5 > 0$

$(-1)^m = 1 > 0$

$\Rightarrow v(0.5, 0.1, 0.4)$ is a local minimum.

c. (Assuming a local optimum is good enough) You should invest 50% in Fund 1, 10% in Fund 2, 40% in Fund 3 to achieve an expected return of 12% at minimum risk (variance).